Ranking Alternative Prospects: A Stochastic Dominance Based Approach

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Abstract

The problem considered here is that of quantifying the extent to which stochastically dominant and dominated distributions differ. For example consider a policymaker's choice between policies when facing a set of distinct, non-combinable options, or consider the comparison of circumstance class outcome distributions in an equality of opportunity study. When policies are not combinable, for example when public investments are "lumpy", and a convex combination of the two options is not feasible, the classic comparative static solution to the choice problem is not available. The approach proposed here is an adaptation of the solution to the problem of choosing the best statistical test amongst a collection of tests, based upon their power function properties. It is supposed that the policies could be combined, and the ideal "stochastically dominant" or optimal envelope of outcomes that could be obtained with such a combination is constructed under a policymaker's given imperative, and then that policy whose outcome most closely approximates this ideal is selected by employing a statistic that measures proximity. The paper concludes with three illustrative examples.

1 Introduction

A cornerstone in the advance of expected utility and prospect based choice theory when considering an aggregate or expected utility objective[1], has been the use of stochastic dominance criteria (modified in the case of prospect theory) to establish the unambiguous superiority of one outcome distribution over another. Predicated upon the nature of preferences, it provides a set of conditions which the preferred outcome distribution associated with a particular state of the world should satisfy relative to its competitor state. The technique has a wide range of applications, yet in spite of a well-developed theory for public policy applications (Lefranc et al. 2008, 2009, and Moyes and Shorrocks 1994), it is seldom used in practice for basically two reasons. Firstly at low orders of dominance, the method only provides an incomplete ordering (comparisons are not always conclusive). Secondly the method does not yield a measure of "by how much one policy (or portfolio) is better than another" for policymakers (or investors) to hang their hats on in the choice process.

These difficulties associated with the practicality of stochastic dominance techniques have long been recognized in the finance literature, where the conventional second order stochastic dominance criterion appropriate for risk averse actors (Rothschild and Stiglitz 1970) was acknowledged by the same authors (Rothschild and Stiglitz 1971) to have no obvious comparative statics properties[2]. That literature responded to this concern with the introduction of an alternative form of dominance, namely "central dominance", characterizing "greater central riskiness" (Gollier 1996). Central Dominance, which is neither stronger nor weaker than second order stochastic dominance, characterizes the necessary and sufficient conditions under which a change in risk changes the optimal value of an agent's decision variable in a predictable fashion for all risk-averse agents[3]. An important feature of this analysis is that the decision variable is continuously related to the risk measure[4], so that incremental changes in the decision variable can be contemplated as a consequence of incremental changes in risk. However, the notion of Central Dominance

[1]Kolm (1966), Atkinson (1987), Foster and Shorrocks (1988), Rothschild and Stiglitz (1970), and Kahneman and Tversky (1979)

[2]They demonstrated that an increase in risk characterized by 2^{nd} order dominance does not necessarily induce all agents to reduce holdings of the risky asset.

[3]Chuang et al. (2013) have developed tests for Central dominance.

[4]The risk free−risky asset mix parameter in the case of the portfolio problem or the tax parameter(s) in a public choice problem.

has not yet found expression in the wellbeing policy literature (for an exception see Chuang et al. (2013)), probably because in many situations policy alternatives are generally not continuously connected in the manner that a convex combination of a risky and risk free asset can be contemplated in the portfolio problem[5]. Rather, policy alternatives are a collection of distinct, non-combinable policies or prospects, and the choice problem is that of picking one of them. In these circumstances, where a variety of alternative policy outcomes is being contemplated (usually in terms of the income distributions they each imply), a collection of pairwise dominance comparisons will have to be made without recourse to the comparative static feature that the notion of central dominance provides.

While much can be learned about the relative status of alternative policy outcomes by considering them under different orders of dominance comparisons, the partial ordering nature of the technique frequently renders the comparisons inconclusive. In fact successive orders of dominance comparison attach increasing importance (weight) to lower values of the income variable in question, so that increasing orders of dominance may be construed as reflecting "successively increasing degrees of concern for the poor" policy imperatives that confront a policymaker. The usual practice in the empirical wellbeing literature is to compare alternative outcome distributions (usually income size distributions) at successive orders of dominance, until dominance at a given order is established. Unfortunately in terms of a collection of pairwise comparisons, this can be a complicated and lengthy process which is frequently impractical (hence the lack of its use).

Here indices are proposed for measuring the extent to which one policy is "better" than another within the context of a specific dominance class, the choice of which reflects the particular imperative confronting the policymaker. Conceptually the index is based upon the approach taken in the statistics literature[6] to choosing from a finite collection of alternative tests, on the basis of the eyeballed proximity of each test's power function to the envelope of the set of available power functions. The envelope reflects the maximal power that could be obtained if the best bits of each test could be notionally combined. So here alternative policy options are considered in the context of a dominance class determined by the policymaker's imperative. The stochastically dominant envelope of policy consequences at the given order of dominance is constructed, so that a measure of the proximity to this envelope for each of the policy options can be calculated. The

[5]Central Dominance could for example be employed in examining a revenue neutral redistributive tax policy, which is some convex combination of lump sum and progressive tax.

[6]See for example Ramsey (1971), Juhl and Xiao (2003), Omelka (2005), and Anderson and Leo (2014).

policy option most proximate to this dominating envelope, i.e. the one with the smallest proximity measure is to be preferred.

In the following, Section 2 outlines the relationship between stochastic dominance criteria, wellbeing classes and the notion that a policymaker may want to make a policy choice in the context of an imperative associated with a particular wellbeing class. In Section 3 the indices appropriate for making such choices are developed. Section 4 exemplifies the technique in three scenarios. Firstly, using a sample of weekly pre-tax incomes drawn from the Canadian Labour Force Survey for January 2012, as the basis for three non-combinable alternative revenue neutral policies from which a choice has to be made. Secondly, in examining the proximity of the attainment outcomes of circumstance classes in an equality of opportunity exercise based on a PISA data set for Germany. Finally, to illustrate the use of the index in a multi-dimensional framework, a comparison of the relative wellbeing of the African Continent with the Rest of The World over the period 1990 -2005 is performed.

2 Stochastic Dominance, Wellbeing Classes and the Policymaker's Imperative

The notion of stochastic dominance was developed as a criteria for choosing between two potential distributions of a random variable x (usually income, consumption or portfolio returns) in order to find the distribution which maximizes $\mathbf{E}(U(x))$ based upon the properties of the function $U(x)$[7], where $U(x)$ represents a felicity function of agents in a society under the income size distribution of x (Levy (1998) provides a summary). The technique yields a decision as to which is the preferred distribution at some specification of $U(x)$, where successively restrictive specifications of $U(x)$ require successively higher orders of dominance comparison. Since dominance at lower orders of comparison implies dominance at higher orders of comparison, the practice is to start comparison at the first order, and making comparison at successively higher orders until an unambiguous decision is reached.

Working with $U(x)$, let x be continuously defined over the domain $[a, b]$, and two alternative states defined by density functions $f(x)$ and $g(x)$ describing the distribution of x across agents in those states. The family of Stochastic Dominance techniques address

[7]Sometimes poorness or poverty indices $P(x)$ are studied (Atkinson 1987) in which case $U(x) = -P(x)$.

the issue: "which state is preferred if the objective is the largest $\mathbf{E}(U(x))$?". Formally, when the derivatives of $U(x)$ are such that $(-1)^{i+1}\frac{d^iU(x)}{dx^i} > 0$, for $i = 1, \ldots, J$, a sufficient condition for:

$$\mathbf{E}_f[U(X)] - \mathbf{E}_g[U(X)] \quad = \quad \int_a^b U(x)(dF - dG) \geq 0 \tag{1}$$

is given by the condition for the dominance of distribution G by F at order $j = 1, 2, \ldots, J$, which is[89]:

$$F_j(x) \leq G_j(x) \;\; \forall \;\; x \in [a,b] \qquad \text{and} \qquad F_j(x) < G_j(x) \;\; \text{for some} \;\; x \in [a,b] \tag{2}$$
$$\text{where :} \;\; F_i(x) = \int_a^x F_{i-1}(z)dz \qquad \text{and} \qquad F_0(x) = f(x)$$

Essentially the condition requires that the functions $F_j(x)$ and $G_j(x)$ not cross, so that the dominating distribution is "unambiguously" below the other. It will be useful for the subsequent discussion to note that $F_i(x)$ (or equivalently $G_i(x)$) may be rewritten in

[8]The intuition for how the result is arrived at is as follows:

For

$$E_f(U(X)) - E_g(U(X)) \quad = \quad \int_a^b U(x)(dF - dG)$$

where $U(x)$ has alternating signed derivatives up to order J with $U' > 0$. Integration by parts yields:

$$E_f(U(X)) - E_g(U(X)) \quad = \quad \text{(Sequence of non-negative terms)} + (-1)^J \int_a^b \frac{\partial^J U(x)}{\partial x^J} \int_a^b \left(G_{J-1}(x) - F_{J-1}(x)\right) dx$$

[9]Formally these are the conditions appropriate for risk averse agents (Levy 1998). If agents are "risk loving", successive derivatives of $U(x)$ would all be positive resulting in conditions of the form:

$$F_J^+(x) \leq G_J^+ \qquad \forall x \in [a,b]$$
$$\text{and} \;\; F_J^+(x) < G_J^+ \qquad \text{for some} \;\; x \in [a,b]$$

where $F_i^+ = \int_x^b F_{i-1}^+(z)dz$, and $F_0^+(x) = f(x)$ These conditions are also used in prospect theory (Levy 1998), and can be examined in the context of the standard conditions for transformed $x^* = -x$ (Anderson 2004). Interestingly in this case successive orders of dominance have the interpretation of increasing concern for the rich since outcomes of the rich get increasing weight in higher order comparisons.

incomplete moment form as:

$$F_i(x) \quad = \quad \frac{1}{(i-1)!}\int_0^x (x-y)^{i-1} dF(y) \tag{3}$$

An important notion regarding dominance relations in what follows is that dominance at order h implies dominance at all orders $h' > h$, and a useful lemma in Davidson and Duclos (2000) is that if F first order dominates G over some region $(-\infty, a)$ then F will dominate G over the whole range of x at some higher order. Thus the practice has been to seek the order at which dominance of one distribution over the other is achieved, for such a comparison is unambiguous at that order of dominance. There is also the implication that at a sufficiently high order of dominance the ordering will be complete rather than partial.

From (2) it may be seen that the dominating distribution is the preferred distribution, reflecting as it does the desire for greater expected $U(x)$. However, as can be seen from (3) increasing orders of dominance attach increasing weight to lower values of x in the population distribution. Thus successively higher orders of dominance can be interpreted as reflecting higher orders of concern for the "poor" end of the distribution. Following Foster and Shorrocks (1988) this permits the interpretation of various forms of dominance as follows[10]:

- $U_{i=1}$, which only requires $\frac{dU(x)}{dx} > 0$, and yields a 1^{st} order dominance rule, is referred to as *Utilitarian* societal preference, and is really an expression of preference for more of x without reference to the spread of x. In the context of the dominance relation, the weight attached to each value of x in the population distribution is the same. However in the cumulative distribution, heuristically the first increment of $f(x)$ is counted at every value of x, the second increment of $f(x)$ is counted at every value of x except the first, the third increment of $f(x)$ is counted at every value of x except the first and second .... In terms of the policymaker's imperative, she would be indifferent to revenue neutral transfers between agents.

- $U_{i=2}$, which requires $\frac{dU(x)}{dx} > 0$, $\frac{d^2U(x)}{dx^2} < 0$, and yields a 2^{nd} order dominance rule, is referred to as *Daltonian* societal preference, and is an expression of preference for

[10]The comparison procedures have been empirically implemented in several ways, see for example Anderson (1996, 2004), Barrett and Donald (2003), Davidson and Duclos (2000), Linton et al. (2005), Knight and Satchell (2008), and McFadden (1989).

more x with weak preference for reduced spread. On the margin, for two distributions with equal means but different variances (whose cumulative distribution will cross, thus contradicting the 1st order dominance criteria), the one with the smallest variance will be preferred. In the context of the dominance relation formula, the weight attached to each value of x in the population distribution decreases as x increases (heuristically the first increment of $f(x)$ is counted twice at every value of x, the second increment of $f(x)$ is counted twice at every value of x except the first, the third increment of $f(x)$ is counted twice at every value of x except the first and second, ..., and in terms of x, equation (3) reveals the increments are units of x). In terms of the policymaker's imperative she would have a preference for revenue neutral transfers from rich agents to poor agents.

- $U_{i=3}$, which requires $\frac{dU(x)}{dx} > 0$, $\frac{d^2U(x)}{dx^2} < 0$ and $\frac{d^3U(x)}{dx^3} > 0$, yields a 3rd order dominance rule, and is an expression of preference for more, with a weak preference for reduced spread especially at the low end of the distribution. In the context of the dominance relation, the weight attached to each value of x in the population distribution decreases at an even faster rate as x increases (heuristically the first increment of $f(x)$ is counted thrice at every value of x, the second increment of $f(x)$ is counted thrice at every value of x except the first, the third increment of $f(x)$ is counted thrice at every value of x except the first and second, ..., and in terms of x, equation (3) reveals the increments are squares of units of x). In terms of the policymaker's imperative she would have a preference for revenue neutral transfers from rich agents to poor agents, and the preference would be stronger the poorer the agent.

  ...

- U_∞ or infinite order dominance is referred to as *Rawlsian* societal preference, since it attaches infinite weight to the poorest individual, and can be examined in the context of the relative incomes of the poorest individuals in two equally populated societies. Essentially the outcome distribution which yields the best outcome for the poorest individual is the one that is chosen.

With this in mind, the policymaker is lead to contemplate a particular order of dominance (choice of i) in order to reflect the imperative she confronts, in terms of the degree of concern for the poorer agents in a society. Thus if the policymaker was indifferent as

to where in the distribution of incomes revenue neutral transfers were made, she would consider 1st Order Dominance comparisons. If on the other hand the policymaker deems it politic to give added weight to the concerns of the poor, policy comparisons should be conducted in terms of higher orders of dominance (values of i greater than 1) of the distributions of policy outcomes. In this context, it may be that no policy dominates at the chosen level of concern. However the policymaker could choose the policy which gets closest to the envelope of alternative policy outcomes, at the appropriate level of integration (i) which reflects the imperative she confronts, if indices of proximity to the envelope were available.

3 The Comparison Indices

Suppose we are to contemplate a collection of wellbeing distributions $G(x)$, $H(x)$, $J(x)$, $\ldots$, $K(x)$, which are the consequence of alternative policy measures, where for convenience $x \in [0, \infty)$. In the context of wellbeing comparisons, we are lead to consider a collection of pairwise comparisons within the family of dominance criteria where j'th order dominance is of the form given in (2) above. Anderson (2004) interpreted dominance between F and G at a particular order as a measure of the degree of separation between the distributions at that order, and the area between the two curves provides a very natural measure of the magnitude of the separation. Furthermore, when G dominates F at the j'th order, and given μ^j is a location measure of x such as the mean, median or modal value of x, note then that:

$$\mathcal{PB}_j \;=\; \frac{-1}{\mu^j} \int_0^\infty [G_j(z) - F_j(z)]\, dz \tag{4}$$

provides a standardized measure of such a separation or wellbeing excess of G over F, where the metric of the unstandardized measure is related to the units of μ^j, making this a unit free measure. However such an index only works if G dominates F at this order. What if there is no dominant policy at a given order?

Suppose the policymaker's imperative is *utilitarian*, it may well be that there is no 1st order dominant policy in the collection. Consider the lower frontier or envelope of all distributions $G(x)$, $H(x)$, $J(x)$, $\ldots$, $K(x)$ in the collection given by $\mathcal{LE}(x) = \min_x\{G(x), H(x), J(x), \ldots, K(x)\}$. Thus in so doing, effectively the "best policy" of each point x has been selected to produce the best possible synthetic policy over the whole

range of x, if all policies could be combined. Obviously $\mathcal{LE}(x)$ would dominate all distributions $G(x)$, $H(x)$, $J(x)$, ..., $K(x)$ at the first order and would thus, if it existed, be the preferred distribution (Note that if one of the distributions in the collection 1st Order Dominated all of the other distributions, $\mathcal{LE}(x)$ would be equal to it). Proximity to such a distribution would be of interest in evaluating each of the available distributions at the first order imperative. Hence we are led to contemplate,

$$\min_{\mathcal{M}(x)} \mathcal{LEPB}(\mathcal{M}(x)) \quad = \quad -\frac{1}{\mu}\int_0^\infty [\mathcal{M}(x) - \mathcal{LE}(x)]dx \tag{5}$$

where $\mathcal{M}(x) = \{G(x), H(x), J(x), \ldots, K(x)\}$, since the lowest value of $\mathcal{LEPB}$ represents the closest proximity to the envelope.

If the policymaker's imperative is represented by the j'th degree dominance criterion, one could contemplate the lower frontier or envelope of all possible j'th order integrals of the candidate distributions, $G_j(x)$, $H_j(x)$, $J_j(x)$, ..., $K_j(x)$, which would be $\mathcal{LE}_j(x) = \min\{G_j(x), H_j(x), J_j(x), \ldots, K_j(x)\}$. Note that $\mathcal{LE}_j(x)$ would stochastically dominate all distributions $G(x)$, $H(x)$, $J(x)$, ..., $K(x)$, at the j'th order and would thus, if it existed, be the preferred distribution at that order[11]. Proximity to such a distribution would be of interest in evaluating the available distributions, hence she would be led to contemplate,

$$\min_{\mathcal{M}_j(x)} \mathcal{LEPB}_j(M(x)) \quad = \quad -\frac{1}{\mu_j}\int_0^\infty [\mathcal{M}_j(x) - \mathcal{LE}_j(x)]dx \tag{6}$$

where $\mathcal{M}_j(x) = \{G_j(x), H_j(x), J_j(x), \ldots, K_j(x)\}$.

The question arises as to whether this statistic possesses the "independence of irrelevant alternatives" property. The answer is yes, in the sense that if one policy is strictly dominated by a combination of all other policies in the collection, then it can be considered irrelevant[12]. By definition of the lower envelope, it will not be present anywhere in the constructed $\mathcal{LE}$, and so will not affect the value of the statistics in any way.

[11] Again note that if one of the distributions j^{th} Order Dominated all of the other distributions $\mathcal{LE}_j(x)$ would be equal to it. It is of interest to note that for j = 1:

$$\int_0^\infty (G(x) - H(x))dx \quad = \quad \int_0^\infty (1 - H(x) - (1 - G(x)))dx = \mu_{H,x} - \mu_{G,x}$$

which can readily be seen to be a difference in means test.

[12] The proof of which is in appendix A.1

Finally, if a statistical comparison of the indices is required, note that the difference between 2 non-normalized indices, $[\mathcal{LEPB}_j(G_j(x)) - \mathcal{LEPB}_j(H_j(x))]$ for example, may be written as:

$$\int_0^\infty [G_j(x) - \mathcal{LE}_j(x)]\,dx - \int_0^\infty [H_j(x) - \mathcal{LE}_j(x)]\,dx \;=\; \int_0^\infty [G_j(x) - H_j(x)]\,dx \qquad (7)$$

which can be estimated, and appropriate inference performed following Davidson and Duclos (2000), details of which are outlined in the appendix A.2.

3.1 A Measure of the Relative Merit of Policies

Consider the upper frontier or envelope of all possible j'th order integrals of the candidate distributions, namely $\mathcal{UE}_j(x) = \max_j\{G_j(x), H_j(x), J_j(x), \ldots, K_j(x)\}$, this constitutes the worst that the policy maker could do if she could combine the policies in a bad way. Proximity to $\mathcal{UE}(x)$ is a measure of how bad the chosen policy is and the area between this theoretical outcome (the upper frontier) and the best theoretical outcome (the lower frontier) given by:

$$S_j \;=\; \int_0^\infty (\mathcal{UE}_j(x) - \mathcal{LE}_j(x))\,dx \qquad (8)$$

constitutes a measure of the range of possibilities available to the policy maker. If it is close to 0 it suggests that it really doesn't matter, all policies are very similar in outcome. A measure of the relative merit of policies $M_j(x) = \{G_j(x), H_j(x), J_j(x) \ldots, K_j(x)\}$ is:

$$\mathcal{PM}_j \;=\; 1 - \frac{\mathcal{LEPB}_j(x)}{S_j} \qquad (9)$$

This would be a number between 0 and 1 and provides a complete ordering of policies at the j'th order of integration or concern for the poor.

4 Three Illustrative Examples

To illustrate these ideas we consider three examples, one from a standard public policy choice literature where the policy maker is confronted with three policy alternatives which yield the same expected return, the other example is drawn from the equality of opportunity literature which has recently employed dominance techniques in examining whether or not the socially just state has been attained. The third is a multidimensional

application measuring the the change in the relative disadvantage the continent of Africa has vis-à-vis the Rest of the World in the joint distribution of of life expectancy and GDP per capita.

4.1 Example 1

In the first, we contemplate three alternative non-combinable policies, A, B, and C that yield the same per capita return in terms of expected post-tax income to society. The different policies have different redistributional effects, which will be characterized through different revenue neutral tax policies on the initial distribution, which shall be denoted policy A. Using the results of Moyes and Shorrocks (1994) it is assumed that the effect of policy B is that of a proportionate tax $t_p(x) = t$, where $0 < t < 1$, whose aggregate proceeds are distributed equally across the population at a level of M per person. The effect of policy C was equivalent to a progressive tax $t_{pr}(x) = t_1 + t_2 F(x)$ (where $F(x)$ is the cumulative distribution of $f(x)$, the income size distribution of pre-tax income x, and $0 < t_1 + t_2 F(x) < 1$, so that $0 < t_1 < 1$, and $0 < t_2 < 1 - t_1$), and again the aggregate per capita proceeds M is distributed equally across the population. All tax regimes are revenue neutral, which implies that the post-tax income for policy B is $(1-t)x + M$, and revenue neutrality implies:

$$\int (tx - M) dF(x) \;\; = \;\; 0 \Rightarrow M = t\,\mathbf{E}(x)) \tag{10}$$

and for policy C, post-tax income will be $(1 - t_1 - t_2 F(x))x + M$, with revenue neutrality implying:

$$\int ((t_1 + t_2 F(x))x - M) dF(x) \;\; = \;\; 0 \Rightarrow t_2 = \frac{(M - t_1\,\mathbf{E}(x))}{(\int x F(x) dF(x))} \tag{11}$$

The empirical analogues of the policies applied to a random sample of n pre-tax weekly incomes x_i, $i = 1, \ldots, n$ (where incomes x are ranked highest 1 to lowest n) drawn from the Canadian Labour Force Survey for January 2012 (wage rate multiplied by usual hours of work per week) would yield post-policy incomes y_i of,

- A: $y_i = x_i$
- B: $y_i = (1-t)x_i + M$
- C: $y_i = \left[1 - t_1 - t_2\left(1 - \frac{\text{rank}(x_i)}{n}\right)\right] x_i + M$

Income distributions that are the result of the three policy alternatives are illustrated in figure 1. The sample size was $52,173$, the parameters were chosen as $t = 0.5$, $t_1 = 0.3$, and as a consequence $t_2 = 0.2976$, and summary statistics for the three policies are presented in Table 1. All three distributions have the same average income with the dispersion ranking $A > B > C$, all are right skewed with Policy C being the least skewed.

Figure 1: Density Functions of Policy Outcomes

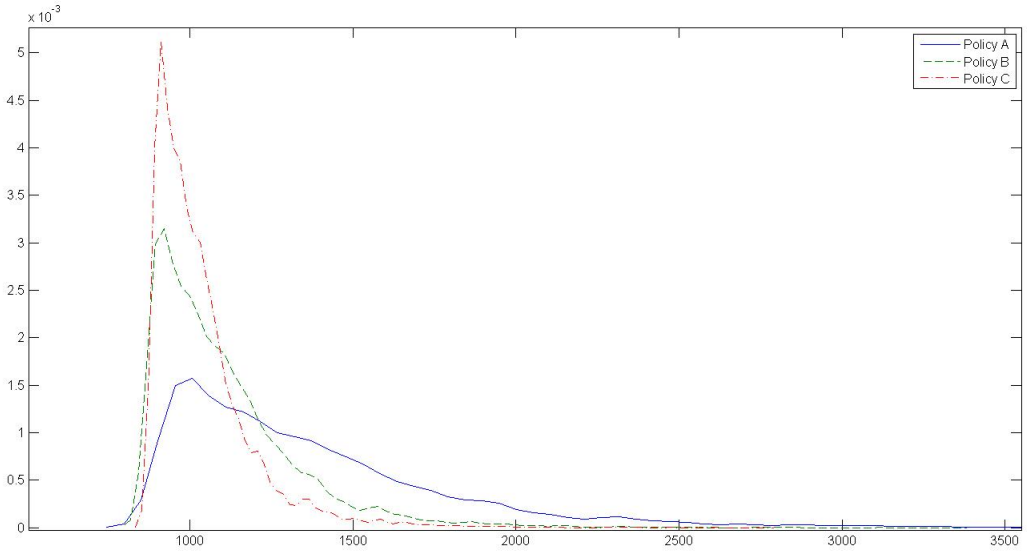


Table 1: Income Distribution Summary Statistics

	Policy A	Policy B	Policy C
Mean Income	836.89	836.89	836.89
Median Income	750.00	793.44	831.73
Standard Deviations	534.41	267.20	205.10
Maximum Income	5769.60	3303.24	2740.27
Minimum Income	4.80	420.84	421.80

Table 2 reports the dominance relationships between the policies in terms of the maximum and minimum differences between the 1^{st}, 2^{nd}, and 3^{rd} orders of integration of the respective distributions (positive maximums together with negative minimums imply no

dominance relationship at that order of integration). As is evident, there are no dominance relationships between the policy outcomes at the 1st order, at the 2nd order A is dominated by both B and C, though there is no dominance relationship between B and C, and at the 3rd order comparison outcome C universally dominates, and will be the envelope of the three distributions at that level of dominance comparison. Note incidentally that if a *Rawlsian*, infinite order dominance, imperative confronted the policymaker, policy C would be the choice since it presents the best outcome for the poorest person. Nonetheless, the primary point here is stochastic dominance's inability to provide a resolution to the policy choice problem should the policymaker's imperative be *utilitarian*, or *Daltonian* in nature.

Table 2: Between Policy Dominance Comparisons ($A \succ_k B$ implies k^{th} order dominance of A over B)

	$A-B$	$A-C$	$B-C$
1st Order minimum difference	-0.1272	-0.1844	-0.0734
1st Order maximum difference	0.2359	0.2527	0.1019
	No Dominance	No Dominance	No Dominance
2nd Order minimum difference	0.0000	0.0000	$-1.9895e^{-12}$
2nd Order maximum difference	0.1225	0.1531	0.0318
	$B \succ_2 A$	$C \succ_2 A$	No Dominance
3rd Order minimum difference	0.0000	0.0000	0.0000
3rd Order maximum difference	0.1529	0.1738	0.0209
	$B \succ_3 A$	$C \succ_3 A$	$C \succ_3 B$

To highlight the merit of this comparison technique, consider the Non-Standardized Policy Evaluation Indices reported in Table 3. Under a *Utilitarian* imperative, Policy A would be chosen (although the magnitudes of each respective policy index suggests that there is very little to choose between the policies at this order of dominance comparison). Under a second order inequality averse imperative, Policy C would be chosen, and under a third order inequality averse imperative, where poorer agents are of greater concern, Policy C would still be chosen (note here the Index is zero because Policy C's distribution, in being uniformly dominant at the third order over the other distributions, will constitute the lower envelope at that order).

Table 3: Policy Evaluation Indices

	Policy A	Policy B	Policy C
$\mathcal{LEPB}_1(\mathcal{M}(x))$	22.2199	22.2359	22.2310
	(0.5018)	(0.5012)	(0.5012)
$\mathcal{LEPB}_2(\mathcal{M}(x))$	25.2379	3.0398	$1.4205e^{-10}$
	(0.0000)	(0.9003)	(1.0000)
$\mathcal{LEPB}_3(\mathcal{M}(x))$	140.9837	17.1928	0.0000
	(0.0000)	(0.8822)	(1.0000)

Note: Efficiency index of (9) are in parenthesis.

4.2 Example 2

In the second example, we turn to the equality of opportunity literature and seek to provide a means of evaluating progress toward an equality of opportunity outcome. Recently dominance techniques have been applied to equality of opportunity analyses, where absence of dominance (usually of 2^{nd} order) of outcome distributions for circumstance classes is seen as indicating equality of opportunity Lefranc et al. (2008, 2009). In this context the increasing orders of dominance reflect an "equality between the outcome distributions" imperative with increasing weight attached to outcomes at the lower end of the distributions "equalizing upward" rather than "equalizing downward".

Using Programme for International Student Assessment (PISA) data for German students in 2003 and 2009, we construct an achievement index following Anderson et al. (2011) (ACL) for students who have completed exams in Language, Math and Science, and a similar ACL circumstance index for each student based upon its family type, parental income, and education. Three equal sized inheritance classes were established based upon the child's circumstance index, and normal distributions were fitted to outcomes of the children in the respective inheritance classes.

From Table 4 note that child outcomes of the higher circumstances classes 1^{st} Order Dominate child outcomes of lower circumstance classes in every case, which rejects the notion of equality of opportunity for both time periods. But the question is, is there a sense that things have improved in Germany over the period 2003-2009? Are we in some sense closer to the equality of opportunity paradigm. Table 5 presents the difference in

Table 4: Summary Statistics of Achievement Index by Inheritance Class

	Class 1 Low Inheritance	Class 2 Middle Inheritance	Class 3 High Inheritance
2003 Mean Achievements	0.0965	0.1692	0.2392
(Standard Deviations)	(0.0300)	(0.0141)	(0.0377)
2009 Mean Achievements	0.0764	0.1232	0.1799
(Standard Deviations)	(0.0204)	(0.0110)	(0.0244)

the indices between the various inheritance classes for 1st, 2nd and 3rd order comparisons for the years 2003 and 2009.

Table 5: Stochastic Dominance Comparison Across Inheritance Class by Year

	Low vs. Middle Inheritance Classes	Low vs. High Inheritance Classes	Middle vs. High Inheritance Classes
2003 1st Order Comparison	0.2049	0.4022	0.1972
2003 2nd Order Comparison	0.0469	0.0753	0.0284
2003 3rd Order Comparison	0.0046	0.0063	0.0018
2009 1st Order Comparison	0.1922	0.4246	0.2324
2009 2nd Order Comparison	0.0306	0.0540	0.0233
2009 3rd Order Comparison	0.0021	0.0031	0.0010

Under 1st order comparisons the results are ambiguous. By comparing the differences across the two years it appears that low inheritors have closed the gap on middle inheritors, however the poor and middle inheritors' gap with respect to upper inheritors had widened. On the other hand, results based on 2nd and 3rd order comparisons are clear, all lower level inheritors have gained on higher level inheritors in every case over the two comparison years. That is to say when the low outcomes in each inheritance class are weighted more heavily, clear advances have been made over the 6 year period.

4.3 Example 3

To illustrate the use of the statistic in a multidimensional framework, equation (4) is used as a wellbeing measure (in essence it computes the volume between the surfaces). Drawing from a convergence-polarization study of African nations versus the rest of the world (Anderson et al. 2012), comparing the GDP per capita and life expectancy of nations over the period 1990-2005, the cumulative joint densities for the two groups of nations for the comparison years are presented in figures 2 through 5. Observe from table 6, that the rest of the world distribution first order stochastically dominates Africa in both periods so that the index corresponds to a measure of Africa's wellbeing deficiency vis-à-vis the rest of the world. As may be observed the deficiency has increased over the period reflecting the fact that Africa and the Rest of the World have polarized.

Table 6: Between Rest of the World & Africa 2 Dimensional 1st Order Dominance Comparisons by Year ($F_{Rest} \succ_1 F_{Africa}$ implies 1st Order Dominance of $Rest$ over $Africa$)

	1990	2005
	$F_{Rest} \succ_1 F_{Africa}$	$F_{Rest} \succ_1 F_{Africa}$
1st Order Minimum Difference	-0.7153	-0.8071
1st Order Maximum Difference	0.0000	0.0022
	$F_{Rest} \succ_1 F_{Africa}$	$F_{Rest} \succ_1 F_{Africa}$
Volume Between the Surfaces	$2.5715e^3$	$3.2933e^3$

Figure 2: 1990 CDF of the African Continent

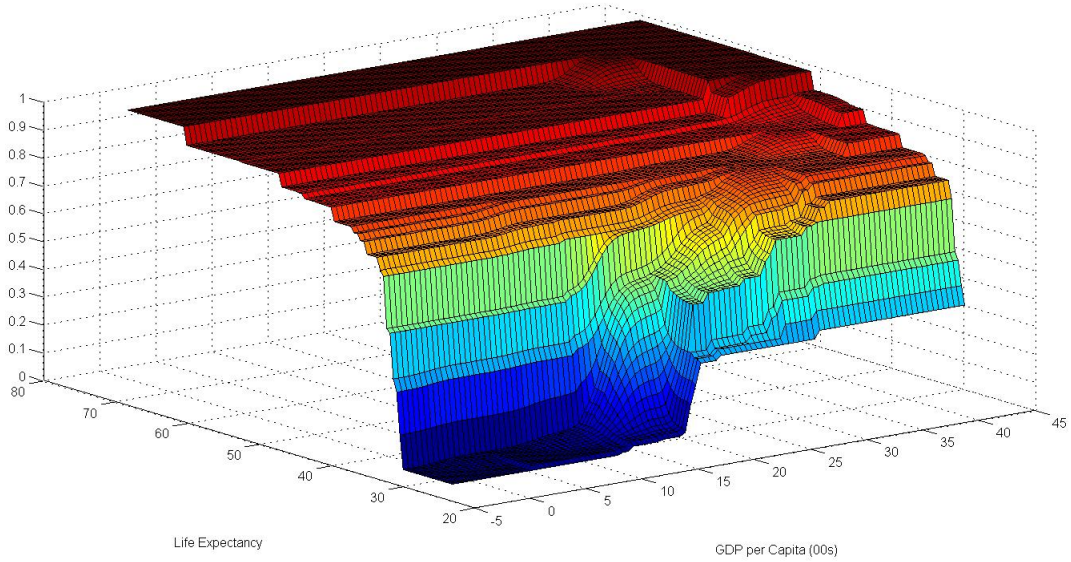


Figure 3: 2005 CDF of the African Continent

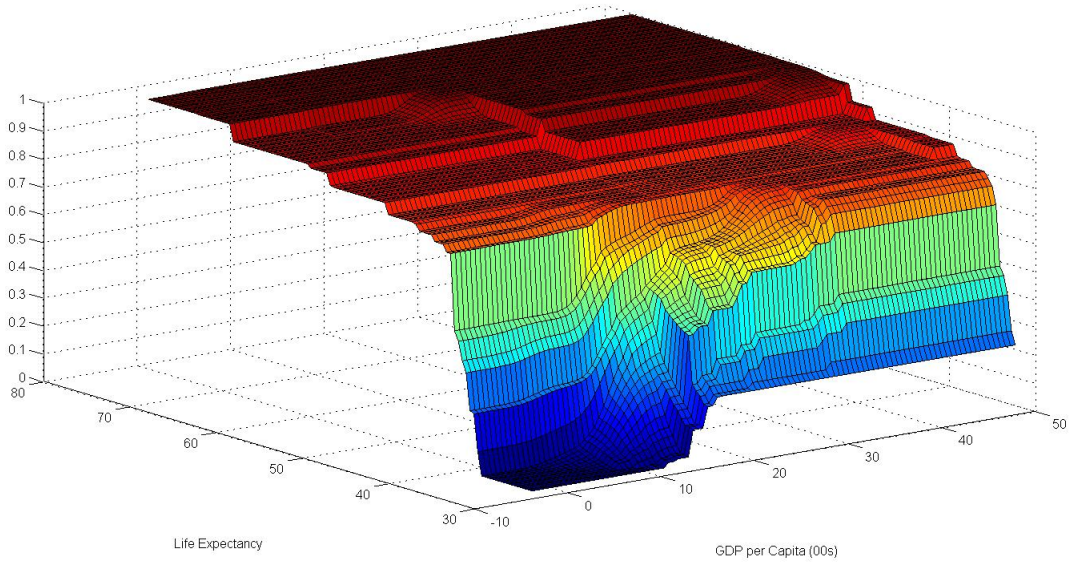


Figure 4: 1990 CDF of All Other Countries

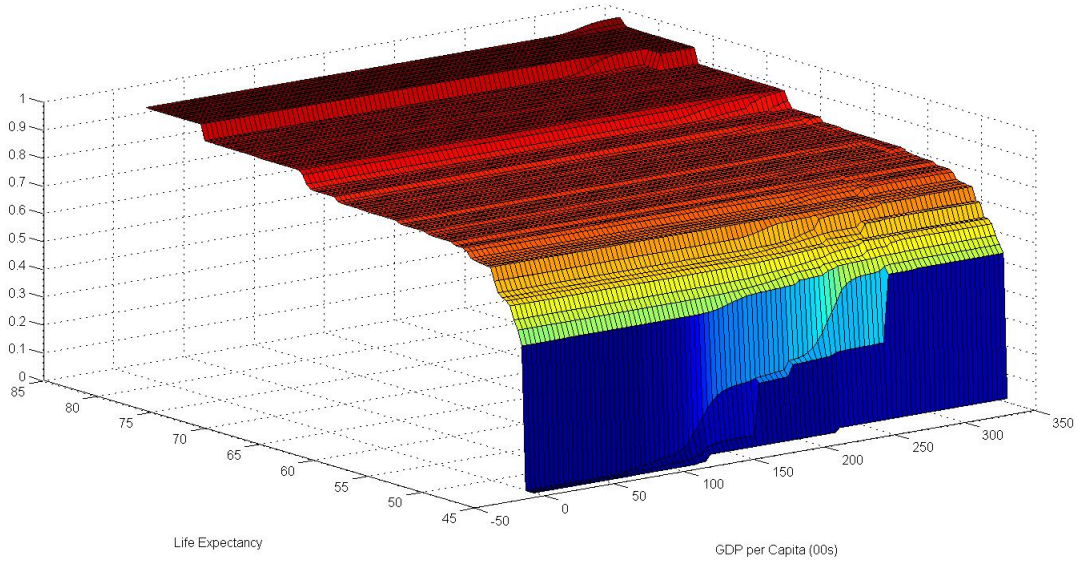
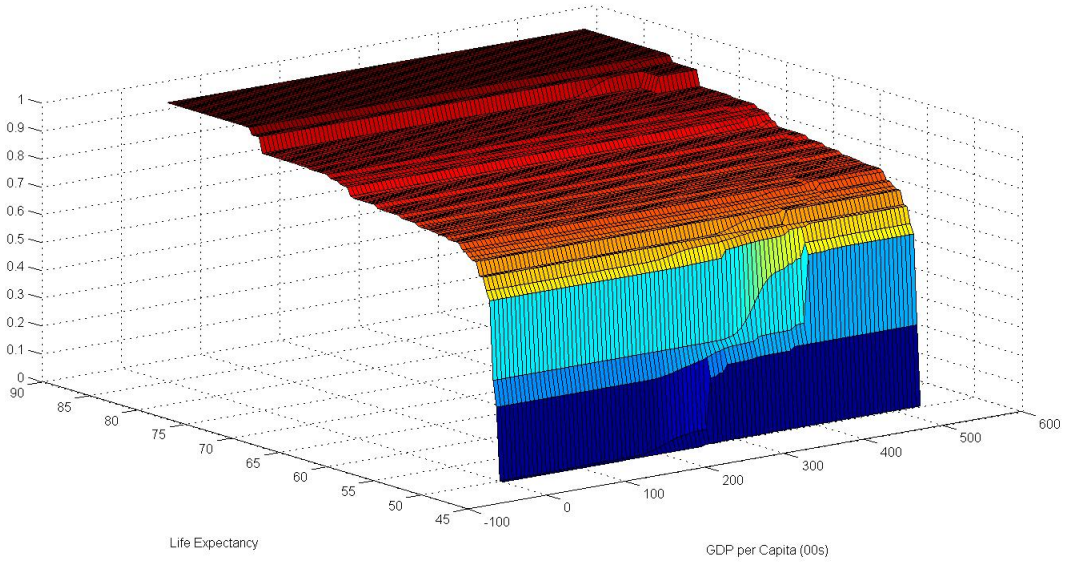


Figure 5: 2005 CDF of All Other Countries



5 Conclusions

The difficulties in applying Stochastic Dominance Techniques in the realm of public policy are twofold. Generally the technique only offers a partial ordering, and furthermore it never yields the policy maker a number by which she can assess "by how much" one policy is better than another. Here measures or indices are proposed which are founded on stochastic dominance principles, and which provide the policy maker with a measure of how much better one policy is than another in the context of the particular distributional imperative she confronts. The use of the statistics is exemplified in three examples.

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A Appendix

A.1 Independence of Irrelevant Alternatives

Here the notion of Independence of Irrelevant Alternatives (IIA) employed is that the introduction of an irrelevant alternative should not change the ordering of all other alternatives. Let the set of policies considered at the j'th level of dominance be

$$\{G_j(x), H_j(x), J_j(x), \ldots, K_j(x)\}$$

and suppose $G_j(x)$ is the irrelevant alternative so the ordering of $H_j(x)$, $J_j(x)$, ..., $K_j(x)$ remains unchanged when G is included from consideration.

Proposition 1 *$G_j(x)$ is an irrelevant alternative if and only if*

$$G_j(x) \geq \min_x\{H_j(x), J_j(x), \ldots, K_j(x)\}$$

with strict inequality holding somewhere[13]

***Necessity**:*

Suppose not, then for $G_j(x) < \min_x\{H_j(x), J_j(x), \ldots, K_j(x)\}$ for some x, it would be the case that $\mathcal{LE}(G_j(x), H_j(x), J_j(x), \ldots, K_j(x)) < \mathcal{LE}(H_j(x), J_j(x), \ldots, K_j(x))$ for some x, and the ordering of $H_j(x)$, $J_j(x)$, ..., $K_j(x)$ could be changed, contradicting the irrelevance condition.

***Sufficiency**:*

Since $G_j(x) \geq \min_x\{H_j(x), J_j(x), \ldots, K_j(x)\}$ $(= \mathcal{LE}(H_j(x), J_j(x), \ldots, K_j(x)))$ for all x, then

$$\begin{aligned}\mathcal{LE}(G_j(x), H_j(x), J_j(x), \ldots, K_j(x)) &= \min_x\{G_j(x), H_j(x), J_j(x), \ldots, K_j(x)\} \\ &= \min_x\{H_j(x), J_j(x), \ldots, K_j(x)\} \\ &= \mathcal{LE}(H_j(x), J_j(x), \ldots, K_j(x)) \ \ \forall x.\end{aligned}$$

And

$$\mathcal{LEPB} = \int_0^\infty (M_j(x) - \mathcal{LE}(x))$$

for $M_j(x) = \{H_j(x), I_j(x), \ldots, K_j(x)\}$ can be computed independently of G leaving the ordering of $H_j(x)$, $J_j(x)$, ..., $K_j(x)$ unchanged.

[13]Essentially, this requires that policy G be stochastically dominated at the j'th order by the lower envelope of all other policies, thus the lower envelope can be formed without resort to G.

Corollary 1 *If G is stochastically dominated at the j'th order by any other single policy from $H_j(x), J_j(x), ..., K_j(x)$, it will be an irrelevant policy at the j'th order.*

Proof. Suppose H dominates G at the j'th order, then since

$$\begin{aligned} H \succ_j G \Rightarrow \min_x\{G_j(x), H_j(x)\} &= H_j(x) \\ \Rightarrow \min_x\{G_j(x), H_j(x), J_j(x), \ldots, K_j(x)\} &= \min_x\{H_j(x), J_j(x), \ldots, K_j(x)\} \end{aligned}$$

■

Corollary 2 *If G is stochastically dominated at the j'th it is an irrelevant policy at all higher orders of Dominance.*

A.2 Estimating difference in $\mathcal{LEPB}$ of Equation 7

Following Davidson and Duclos (2000), the i'th order stochastic dominance criteria are based upon (3) which may be estimated from a random sample of y's by:

$$\widehat{F}_i(x) = \frac{1}{N(i-1)!}\sum_{j=1}^{N}(x-y_j)^{i-1}\mathbb{I}(y_j < x)$$

Here $\mathbb{I}(s)$ is the indicator function which equals one if s is true, and zero otherwise. For a sequence of values $x_1, x_2, \ldots, x_K$, the estimates of the vector $[F_i(x_1), F_i(x_2), \ldots, F_i(x_K)]'$ can be shown to be asymptotically normally distributed i.e.

$$\begin{pmatrix} \widehat{F}_i(x_1) \\ \widehat{F}_i(x_2) \\ \vdots \\ \widehat{F}_i(x_K) \end{pmatrix} \sim N\left(\begin{pmatrix} F_i(x_1) \\ F_i(x_2) \\ \vdots \\ F_i(x_K) \end{pmatrix}, \begin{pmatrix} C_i(x_1,x_1) & C_i(x_1,x_2) & \ldots & C_i(x_1,x_K) \\ C_i(x_2,x_1) & C_i(x_2,x_2) & \ldots & C_i(x_1,x_K) \\ \vdots & \vdots & \ddots & \vdots \\ C_k(x_K,x_1) & C_i(x_K,x_2) & \ldots & C_i(x_K,x_K) \end{pmatrix}\right)$$

where the covariance terms:

$$C_i(x_j,x_k) = \mathbf{E}\left(\left(\widehat{F}_i(x_j) - F_i(x_j)\right)\left(\widehat{F}_i(x_k) - F_i(x_k)\right)\right)$$

for $j,k = \{1,2,\ldots,K\}$, may be estimated as:

$$\widehat{C}_i(x_j,x_k) = \frac{1}{[N(i-1)!]^2}\sum_{n=1}^{N}(x-y_n)^{i-1}\mathbb{I}(y_n \le x)(z-y_n)^{i-1}\mathbb{I}(y_n \le z) - N^{-1}\widehat{F}_i(x)\widehat{F}_i(z)$$

When distributions $f(.)$ and $g(.)$ are independently sampled, interest centers on the vector of differences $\mathbf{F} - \mathbf{G} = [F_i(x_1) - G_i(x_1), F_i(x_2) - G_i(x_2), \ldots, F_i(x_k) - G_i(x_k)]'$, which under the null of equality is jointly distributed as $N(0, C_{i,f} + C_{i,g})$, where $C_{i,f}$ and $C_{i,g}$ are respectively the covariance matrices under f and g[14].

To examine (7), letting $x >$ maximal value in the pooled sample, the last component of the vector $F_{j+1}(x) - G_{j+1}(x)$, and its corresponding variance estimate would be used for inference purposes.

[14]If f and g were sampled in a panel, then the between distribution covariances also need to be included in the calculus.