More on multidimensional, intertemporal and chronic poverty orderings

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Abstract

As noted by Bourguignon and Fields (1997), being poor can generate significant well-being losses, whatever the extent of observed deprivations, so that marginal changes that lift individuals out of poverty can result in non-marginal variations of poverty and well-being. This paper investigates the consequences of such discontinuities for obtaining robust ordering of intertemporal or multidimensional poverty and contrasts the corresponding stochastic dominance conditions with the one suggested in Duclos, Sahn and Younger (2006) and Bresson and Duclos (2012) that were proposed for classes of poverty indices that show continuity at the frontier of the poverty domain. We prove that the necessary and sufficient conditions for testing the robustness of poverty orderings with respect to the poverty frontier and the poverty index result in an impossibility of testing this robustness from a practical point of view. However, robustness checks can be performed when restrictions are added on the shape or on the location of the poverty frontier. We also show how the suggested tools can help study chronic poverty, the chronic poor being those in the bottom part of the poverty domain (see for instance Jalan and Ravallion, 2000, Foster, 2009).

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[‡] Please note that this version is an alpha release. It is incomplete and may contain notable mistakes and typos. As a consequence, it is likely to be slightly modified in the future.

1 Introduction

Measuring poverty is a social evaluation exercise and consequently relies on value judgements that can rarely exhibit in a unanimous agreement. For instance, multidimensional and intertemporal poverty measurement has resulted in the production of numerous indices,[1] but the availability of such a large set of indices is puzzling for both applied studies and policy making since there are often no obvious reasons for necessarily preferring one index over another. A similar issue arises regarding the definition of the poverty domain. Should it be defined with respect to dimension-specific or period-specific thresholds and then used for a "counting approach"? Or should a more general approach, akin to the use of utility indifference curves be used instead?

In the traditional case of snapshot monetary poverty, robustness to the choice both of poverty indices and of poverty lines has been addressed using the stochastic dominance procedures proposed *inter alia* by Atkinson (1987), Foster and Shorrocks (1988*a*,*b*), Zheng (1999) and Duclos and Makdissi (2004). In the context of multidimensional and intertemporal measurement, dominance procedures have also been proposed by Bourguignon and Chakravarty (2002), Duclos et al. (2006), Yalonetzky (2011), Bresson and Duclos (2012), Yalonetzky (2013) but some of these studies have relied on a continuity axiom that states that poverty indices should be continuous over the whole welfare domain, although many of the commonly used indices exhibit discontinuities, especially at the poverty frontier.[2]

Here, we try to address this issue by considering classes of poverty indices that comply with restricted continuity. Restricted continuity implies that poverty indices are continuous over the poverty domain but can show discontinuities at the frontier of the poverty domain. This axiom is less restrictive than the alternative continuity axiom (see Zheng, 1997) that imposes the poverty index to be continuous over the whole well-being space, and so make it possible to consider broader sets of poverty indices. For instance, for both multidimensional and unidimensional poverty measurement, headcount indices are indeed famous illustrations of poverty indices that satisfy restricted continuity but not continuity. Restricted continuity is also consistent with the idea, investigated by Bourguignon and Fields (1997) for snapshot poverty analysis, that being poor may generate significant well-being losses, whatever the extent of observed deprivations, so that marginal changes that lift individuals out of poverty may result in substantial poverty variations. Though this behaviour makes poverty indices more sensitive to measurement errors, it can reasonably be supported on the basis of ethical arguments. It can notably be suggested that, considering the monetary dimension of poverty, having a income level that reaches 99.9% of the expenditure level that is regarded as necessary for living a decent life is not like having the whole amount that would make it possible for individuals to meet physical, physiological and social primary needs because of indivisibilities

[1] For multidimensional poverty, note in particular Chakravarty, Mukherjee and Ranade (1998), Tsui (2002), Bourguignon and Chakravarty (2003), Chakravarty, Deutsch and Silber (2008), Alkire and Foster (2011) and Chakravarty and d'Ambrosio (2013); for intertemporal poverty, consider Calvo and Dercon (2009), Foster (2009), Hoy and Zheng (2011), Hoy, Thompson and Zheng (2012), Bossert, Chakravarty and d'Ambrosio (2012), Busetta and Mendola (2012), Canto, Grandin and del Rio (2012), Zheng (2012) as well as Dutta, Roope and Zank (2013).

[2] Yalonetzky (2011) addresses the issue of discontinuity but his dominance criteria only apply to Alkire and Foster's (2011) family of poverty indices.

in consumption for instance.

It is also worth considering the existence of discontinuities within the poverty domain when focusing on chronic poverty. The literature proposes various definitions for the concept that also make it desirable to consider the issue of the ethical robustness of chronic poverty ordering. In the present paper we assume that intertemporal poverty indices can be additively decomposed into chronic and transient components, the chronic component being defined on the bottom part of the poverty domain. Putting forward that the chronic component of poverty shares inherits the same basic properties as the one we impose for intertemporal poverty indices, it then can easily be understood that testing the robustness of chronic poverty orderings is not different from testing the robustness of intertemporal poverty orderings from a technical point of view with our discontinuous framework.

The present paper thus proposes dominance conditions so as to check the ethical robustness of multidimensional, intertemporal, or chronic poverty orderings with respect to both the definition of the poverty or chronic poverty frontier and the functional form used for the poverty index. We show that simple stochastic dominance conditions make it possible to extend the one suggested in Duclos et al. (2006) and Bresson and Duclos (2012) for continuous indices. Section 2 introduces concepts and notations. Our main results are presented in section 3 and section 4 show how this results can be extended to chronic poverty orderings. Section 5 concludes.

2 Notations and preliminary concepts

Let overall well-being be a function of two indicators, x_1 and x_2, and be given by $\lambda(x_1, x_2)$.[3] This function is a member of $\mathcal{L}$, defined as the set of continuous and non-decreasing functions of x_1 and x_2. For our purposes, we can think of x_t as income at time t, but intertemporal poverty can also be thought in non-monetary terms so that x_t could also be an index of durable good possessions, an health status, or a social inclusion index at time t. The vector (x_1, x_2) is called an income profile. Without loss of generality, we assume that the variables x_1 and x_2 are defined on the set of positive real numbers, so that $\lambda : \Re^2_+ \to \Re$.

Similarly to Duclos et al. (2006), we assume that an unknown poverty frontier helps distinguishing between the poor and the rich members of the population. We can think of this frontier as a set of points at which the well-being of an individual is precisely equal to a "poverty level" of well-being, and below which individuals are in poverty. This frontier is assumed to be defined implicitly by a locus $\Lambda(\lambda) \in \Re^2_+$ defined by equation $\lambda(x_1, x_2) = 0$, and is analogous to the usual downward-sloping indifference curves in the (x_1, x_2) space. Intertemporal poverty is then defined by states in which $\lambda(x_1, x_2) \leq 0$, and the poverty domain Γ is consequently obtained as:

$$\Gamma(\lambda) := \left\{(x_1, x_2) \in \Re^2_+ | \lambda(x_1, x_2) \leq 0\right\}. \tag{1}$$

Let the joint cumulative distribution function of x_1 and x_2 be denoted by $F(x_1, x_2)$. For analytical simplicity, we focus on classes of additive bidimensional poverty indices, which are

[3] For expositional simplicity, we focus on the case of two dimensions of individual well-being.

the kernels of broader classes of subgroup-consistent bidimensional poverty indices.[4] Such bidimensional indices can be defined generally as $P(\lambda)$:

$$P(\lambda) = \iint_{\Gamma(\lambda)} \pi(x_1, x_2; \lambda)\, dF(x_1, x_2), \tag{2}$$

where $\pi(x_1, x_2; \lambda)$ is the contribution to overall poverty of an individual whose income at period 1 and 2 is respectively x_1 and x_2. The well-known "focus axiom" entails that someone contributes to poverty only if his income profile is in the poverty domain, that is:

$$\pi(x_1, x_2; \lambda) \begin{cases} \geq 0 & \text{if } (x_1, x_2) \in \Gamma(\lambda), \\ = 0 & \text{otherwise.} \end{cases} \tag{3}$$

It deserves to be noted that our approach for intertemporal poverty measurement is very general. More specifically, it is neutral with respect to the issue of aggregation at the individual level. A first approach consists in gathering all elements of the income profile into a unique equivalent income that is then compared with a standard of living. The equivalent income can be for instance permanent income (Rodgers and Rodgers, 1993) or an equally distributed equivalent income (Hoy et al., 2012) that is supposed to yield the level of well-being if earned at each date within the period of interest. A second approach is based on the opposite order for aggregation. Income at each date are compared with the corresponding living standard so as to get deprivation indices and these values are then blended together for each individual into a single index that assesses its level of intertemporal poverty (Foster, 2009, Duclos, Araar and Giles, 2010).

In the next pages, the power of the dominance criteria proposed for measures of the type in (2) is increased by adding assumptions on the poverty effects of income changes at each time period. For this, it is useful to distinguish between profiles with a lower first-period income and profiles with a lower second-period income. The poverty domain can be split into $\Gamma_1(\lambda) := \{(x_1, x_2) \in \Gamma(\lambda) | x_1 < x_2\}$, the set of poverty profiles whose minimal income is found in the first period, and $\Gamma_2(\lambda) := \{(x_1, x_2) \in \Gamma(\lambda) | x_1 \geq x_2\}$, the set of poverty profiles whose minimal income is found in the second period. Equation (2) can then be written as:

$$P(\lambda) = \iint_{\Gamma_1(\lambda)} \pi(x_1, x_2; \lambda)\, dF(x_1, x_2) + \iint_{\Gamma_2(\lambda)} \pi(x_1, x_2; \lambda)\, dF(x_1, x_2), \tag{4}$$

that is, the sum of relatively low-x_1 poverty and of relatively low-x_2 poverty.

For ease of exposition, let the derivatives of π be defined as:

- $\pi^{(i)}(a, b)$, $i = 1, 2$, for the first-order derivative of π with respect to its ith argument,
- and as $\pi^{(a)}(a, b)$, for the first-order derivative of π with respect to the variable a, so that $\pi^{(u)}\big(a(u), b(u)\big) = \pi^{(1)}\big(a(u), b(u)\big)\frac{\partial a}{\partial u} + \pi^{(2)}\big(a(u), b(u)\big)\frac{\partial b}{\partial u}$.

[4]For the unidimensional case, see Foster and Shorrocks (1991).

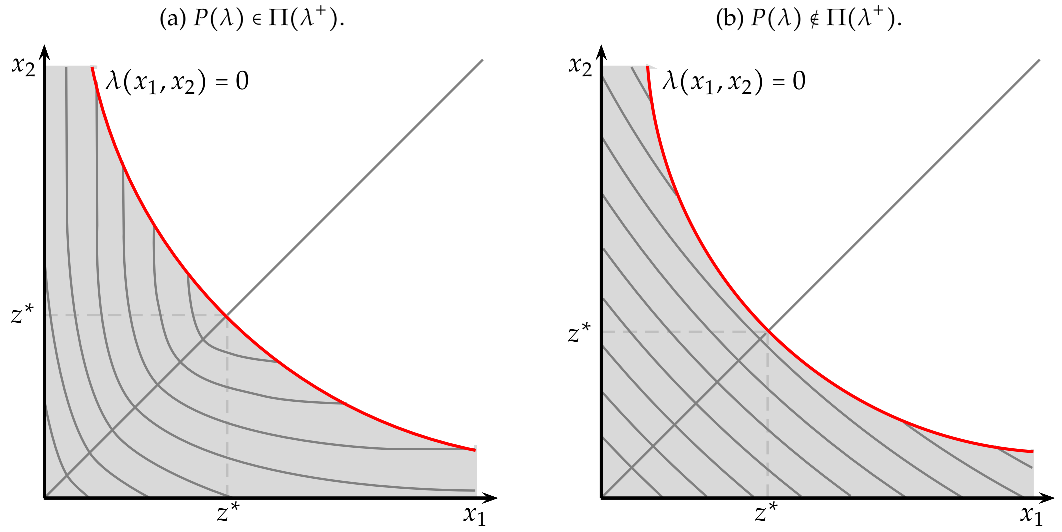

Figure 1: Iso-poverty lines with restricted continuity.

Then, define the class $\Pi(\lambda^+)$ of monotone poverty indices $P(\lambda)$ as:

$$\Pi(\lambda^+) = \left\{ P(\lambda) \;\middle|\; \begin{array}{l} \Gamma(\lambda) \subseteq \Gamma(\lambda^+), \\ \pi(x_1,x_2;\lambda) \geq 0, \text{ whenever } \lambda(x_1,x_2)=0, \\ \pi^{(x_1)}\big(x_1,z_2(x_1)\big) \leq 0, \ \forall x_1 \in [0,z^*], \text{ and } \pi^{(x_2)}\big(z_1(x_2),x_2\big) \leq 0, \ \forall x_2 \in [0,z^*], \\ \pi^{(1)}(x_1,x_2;\lambda) \leq 0 \text{ and } \pi^{(2)}(x_1,x_2;\lambda) \leq 0 \ \forall x_1,x_2, \\ \pi^{(1,2)}(x_1,x_2;\lambda) \geq 0, \ \forall x_1,x_2 \end{array} \right\} \tag{5}$$

where $z_1(x_2)$ and $z_2(x_1)$ are respectively the value of the first and second-period income, such that $\lambda\big(x_1,z_2(x_1)\big)=0$ and $\lambda\big(z_1(x_2),x_2\big)=0$, and z^* is the value of income such that $\lambda(z^*,z^*)=0$.

The class $\Pi(\lambda^+)$ is broader than the class $\ddot{\Pi}(\lambda^+)$ used in Duclos et al. (2006) and Bresson and Duclos (2012) since these study assume $\pi(x_1,x_2;\lambda)=0$ at the frontier of the poverty level. $\Pi(\lambda^+)$ includes *inter alia* the families of bidimensional poverty indices proposed by Chakravarty et al. (1998), Tsui (2002), Chakravarty et al. (2008), Hoy and Zheng (2011), Zheng (2012), and Dutta et al. (2013), as well as some members of the family introduced by Bourguignon and Chakravarty (2003). It also includes the bidimensional headcount index $H(\lambda)$ that indicates the share of the population whose income profile lies within $\Gamma(\lambda)$.

The first condition in equation (5) indicates that the poverty domain for each member $P(\lambda)$ should lie within the poverty domain defined by the well-being function λ^+ (λ^+ then representing the maximum admissible poverty frontier). The second and third conditions conditions in equation (5) relax the usual assumption of continuity at the frontier of the poverty domain. More specifically, the second condition states that poverty can be non-zero at the frontier of the poverty domain. Continuity is often assumed in order to prevent small measurement errors nearby the poverty frontier from resulting in non-marginal variations of the poverty index, but as noted by Bourguignon and Fields (1997) this hypothesis cannot be defended if there is a social cost of being poor, whatever the extent of deprivations, hence implying that there is a non-negligible difference between those at the frontier of the poverty domain and those just above. The third condition means that poverty decreases as we move along the frontier of the poverty domain towards the income profile (z^*, z^*). Figure 1 illustrates the how this condition is likely to influence the shape of iso-poverty curves. While the set of iso-poverty curves shown in the left panel is compatible with poverty indices from $\Pi(\lambda^+)$, this is clearly not the case with the set illustrated on the right panel. Indeed, if we assume as usual—this property results from the fourth condition—that moving to a higher iso-poverty curve result in a lower poverty level, then Figure 1b shows a poverty index such that moving in the direction of profile (z^*, z^*) along the poverty frontier increases the observed level of poverty.

The fourth condition in (5) corresponds to the traditional monotonicity axiom, *i.e.*, that an income increment in any period should never increase poverty.[5] In association with the third condition, it means that $\pi(z^*, z^*) = \min\{\pi(x_1, x_2) | (x_1, x_2) \in \Gamma(\lambda)\}$.

Finally, the last condition in (5) captures the axiom of non-decreasing poverty after a "correlation increasing switch", an axiom introduced by Atkinson and Bourguignon (1982). It is then supposed that the marginal benefit of an income increment at period 1 decreases with the income level at period 2 (and vice versa). Intuitively, this property also says that a permutation of the incomes of two poor individuals during a given period should not decrease poverty if one of them becomes more deprived than the other in both periods.

3 Dominance conditions

3.1 General results

We first introduce what we regard as our core result.

Theorem 1.

$$P_A(\lambda) \geq P_B(\lambda), \ \forall P(\lambda) \in \Pi(\lambda^+), \tag{6}$$

$$\textit{iff} \qquad H_A(\lambda) \geq H_B(\lambda) \ \forall \lambda \in \mathcal{L} \ s.\ t.\ \Gamma(\lambda) \subseteq \Gamma(\lambda^+). \tag{7}$$

Proof. See Appendix A. □

Theorem 1 says that poverty is unambiguously larger for population A than for population B for all poverty sets within $\Gamma(\lambda^+)$ and for all members of the class of bidimensional poverty measures $\Pi(\lambda^+)$ if and only if the bidimensional poverty headcount $H(\lambda)$, including the "intersection" headcount that corresponds to the joint cumulative distribution function, is greater in A than in B for all potential poverty frontiers in $\Gamma(\lambda^+)$. In the case of continuous poverty indices, the ordering power is likely to increase dramatically as condition (7) boils down to the comparisons of "intersection" headcount indices within $\Gamma(\lambda^+)$ (Duclos et al., 2006). Relaxing the continuity assumption extends the dominance condition to an infinite set of bidimensional headcount indices and thus makes robustness checks more demanding. Theorem 1 can be regarded as the multidimensional counterpart of Proposition 1 in Zheng (1999) that extends Atkinson's (1987) first-order dominance result to snapshot poverty indices that fulfil restricted continuity.

[5] As noted in Duclos et al. (2006), we must also have that $\pi^{(1)} < 0$, $\pi^{(2)} < 0$, and $\pi^{(1,2)} > 0$ over some ranges of x_1 and x_2 for the indices to be non-degenerate.

Theorem 1 imposes limited requirements regarding the location of the poverty frontier as we only assume it to be somewhere below λ^+. However, we may also assume that there is also some lower bound λ^- for the frontier of poverty domain, so that the "good" poverty frontier should be located somewhere within $\bar{\Gamma}(\lambda^+,\lambda^-) := \Gamma(\lambda^+) \not\cap \Gamma(\lambda^-)$. In the next paragraphs, we show how the ordering power can be increased by considering a lower bound for the poverty domain.

For this purpose it is first necessary to consider specific forms for headcount indices. As in (Duclos et al., 2006), bidimensional stochastic dominance surface can now be defined using:

$$P^{\alpha,\beta}(z_u,z_v) := \int_0^{z_u}\int_0^{z_v} (z_u-u)^{\alpha-1}(z_v-v)^{\beta-1}\,dF(u,v). \tag{8}$$

where α and β refer to the dominance order in each dimension. Since the present paper focusses on first-order dominance, α and β are set equal to 1. The function $P^{1,1}(z_u,z_v)$ is the "intersection" bidimensional poverty headcount index.

Let also the partial headcount indices $F_{\lambda,1}$ and $F_{\lambda,2}$ be defined by:

$$F_{\lambda,1}(x_1) := \int_0^{x_1} F\big(z_2(y_1)|y_1\big)f_1(y_1)\,dy_1, \tag{9}$$

$$F_{\lambda,2}(x_2) := \int_0^{x_2} F\big(z_1(y_2)|y_2\big)f_2(y_2)\,dy_2. \tag{10}$$

where f_t is the marginal pdf of period t's income and $F(x_t|x_x)$ is the conditional cdf of period t's income given income at period $s \neq t$. The index $F_{\lambda,1}(a)$ ($F_{\lambda,2}(a)$) simply returns the share of the population that is regarded as poor given λ and whose first-period (second-period) income is below a. Consequently, from a technical point of view, it can also be read as a bidimensional headcount index H using a slightly different poverty frontier.

Theorem 2.

$$P_A(\lambda) \geq P_B(\lambda),\ \forall P(\lambda) \in \Pi(\lambda^+),\ \lambda \in \mathcal{L}\ s.\ t.\ \Lambda(\lambda) \subseteq \bar{\Gamma}(\lambda^+,\lambda^-), \tag{11}$$

$$\begin{aligned}
&\textit{iff} && P_A^{1,1}(x_1,x_2) \geq P_B^{1,1}(x_1,x_2),\ \forall (x_1,x_2) \in \Gamma(\lambda^+), && (12)\\
&\textit{and} && H_A(\lambda) \geq H_B(\lambda)\ \forall \lambda \in \mathcal{L}\ s.\ t.\ \Lambda(\lambda) \subseteq \bar{\Gamma}(\lambda^+,\lambda^-), && (13)\\
&\textit{and} && F^A_{\lambda,t}(x) \geq F^B_{\lambda,t}(x) \forall t \in \{1,2\},\ x \in [0,z^*],\ \lambda \in \mathcal{L}\ s.\ t.\ \Lambda(\lambda) \subseteq \bar{\Gamma}(\lambda^+,\lambda^-). && (14)
\end{aligned}$$

Proof. See Appendix A. □

Theorem 2 states that robust poverty ordering with respect to the choice of the functional form for the poverty index and with respect to the poverty frontier, assuming this frontier lies somewhere between two non-intersecting bounding frontiers, can be obtained if three types of conditions are met. Condition (12) is the traditional bidimensional first-order dominance condition over the poverty domain. Condition (13) adds that the share of the population within the poverty domain should not be larger for population B when compared with population A, while condition (14) extends this condition for partial headcount indices $F_{\lambda,1}(z)$ and $F_{\lambda,2}(z)$ for all z below z^*. It can easily be shown from the demonstration (Appendix A) that condition (14) is not necessary if we assume the poverty index π to be constant at the poverty frontier, so that

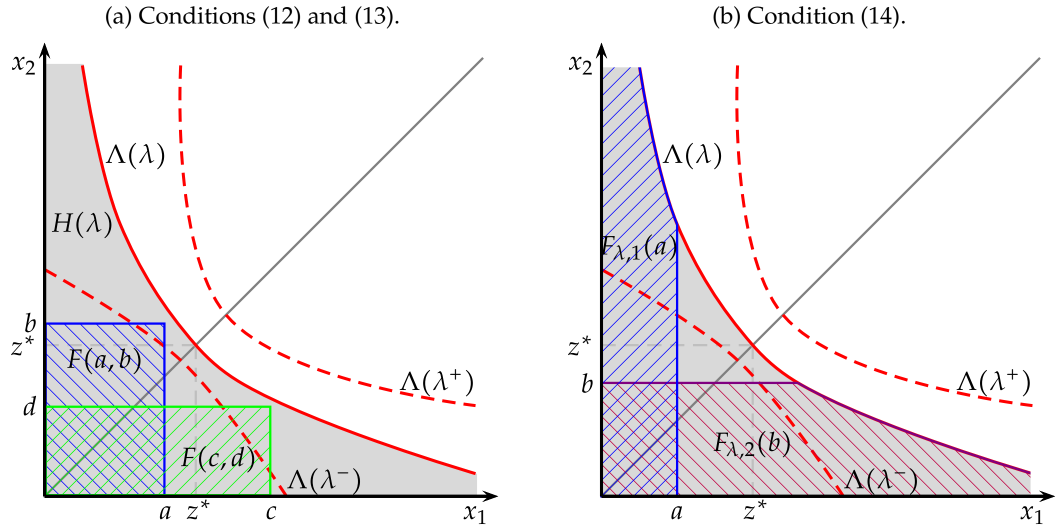

Figure 2: The dominance conditions in Theorem 2 for a given poverty frontier.

$\pi^{(x)}\big(z_1(x), x\big) = \pi^{(x)}\big(x, z_2(x)\big) = 0\ \forall x.$

Figure 2 illustrates the three dominance conditions for a given function λ. For condition (12) to hold, it is necessary to check whether the share of the population for each rectangular area within $\Gamma(\lambda)$ is nowhere larger for population B when compared with population A. Condition (13) simply entails checking whether the share of the population within the grey area is not larger for distribution B than for distribution A. Finally, with condition (14) it necessary for the share of the population within each hatched area defined with respect to points of the frontier of poverty to be nowhere larger for population B when compared with population A.

Although Theorem 2 does not exhibit the same formal elegance in comparison with Theorem 1, Theorem 2 is less demanding since the comparison $H_A(\lambda) \geq H_B(\lambda)$ has not to be performed over the whole maximum poverty domain $\Gamma(\lambda^+)$. For instance, headcount comparisons have to be also performed in $\Gamma(\lambda^-)$, but using uniquely the "intersection" headcount index $P^{1,1}$, hence reducing the computing load and making it possible to rank a larger number of distributions.

The difference between the two results deserves to be stressed since it marks a noticeable difference with respect to traditional snapshot poverty analysis. Indeed, with unidimensional poverty orderings, assuming that there is a minimal value that is strictly larger than the lower bound of the poverty domain does not increase the ordering power at the first order as comparisons of the univariate cumulative distribution function has also to be performed over the whole maximum poverty domain. That change can easily be understood as, for a given value z, there is only one way of defining the headcount index while in most cases there is infinite possibilities for each income profile (z_1, z_2) in the bidimensional case.

In fact, from a practical point of view, Theorem 1 (and Theorem 2 as long as $\bar{\Gamma}(\lambda^+, \lambda^-) \to \varnothing$) can be regarded as an impossibility theorem because we our framework allow the use of an infinite number of definitions for bidimensional headcount indices. In other words, implementing the dominance conditions may be at least difficult, if not virtually impossible. This also contrasts with the first-order dominance conditions in the unidimensional case where the headcount condition does not preclude applicability since there is only a single manner of defining a headcount index with snapshot poverty for a given poverty line. We will nevertheless show in the next paragraphs that the conditions for robust intertemporal poverty orderings become more tractable once we impose additional restrictions regarding the frontier of the poverty domain.

3.2 Some specific cases

A straightforward way to circumvent the issue of implementation is of course to assume that the poverty frontier is known since conditions (13) and (14) have only to be checked with respect to the poverty frontier. However, this option may be regarded as a disappointing solution since it means giving up performing robustness checks with respect to the poverty frontier.

Nevertheless, it is worth noting that only a limited set of functional forms for the poverty frontier have been proposed up to now for either multidimensional or intertemporal poverty measurement, so that it could be reasonable to focus on a subset of $\Pi(\lambda^+)$ that corresponds to some widely used families of poverty frontier. Of course, we do not argue that the die has been cast regarding the potential shape of the poverty frontier so that we should definitively stick to the existing choices but we can admit that highly twisted poverty frontiers—remember that continuity and non-increasingness are the sole requirements for the frontier!—would surely not top the charts as candidates for the "good" poverty frontier. For instance, considering for instance the specific case of intertemporal monetary poverty measurement, it can be presumed that most social researchers would agree that the poverty frontier should be convex.

As a matter of fact, a quick review of existing poverty indices shows that a prominent part of them are based on the "union" definition of the poverty domain, *i.e.* individuals are poor whenever one of their income during the period of interest falls below some threshold z_t ($\Gamma(\lambda) = \left\{(x_1, x_2) \in \mathfrak{R}_+^2 | \min\left\{\frac{x_1}{z_1}, \frac{x_2}{z_2}\right\} \leq 1\right\}$). In this particular case, condition (13) can easily be checked since for each income profile within $\bar{\Gamma}(\lambda^+, \lambda^-)$ we just have to estimate the headcount difference between the two distributions to be compared using "union" headcount indices.[6] Regarding, condition (14), the test is also easy to implement since the functions $F_{\lambda,1}(x_1)$ and $F_{\lambda,2}(x_2)$ take simple functional form. Indeed, we obtain for each "union" poverty domain defined with respect to the profile $(z_1, z_2) \in \bar{\Gamma}(\lambda^+, \lambda^-)$:

$$F_{\lambda,1}(x_1) = \begin{cases} F(x_1) & \text{if } x_1 \leq \min\{z_1, z_2\} \\ F(z_1) + P^{1,1}(x_1, z_2) - P^{1,1}(z_1, z_2) & \text{otherwise} \end{cases}, \tag{15}$$

[6] The condition is reminiscent of dominance conditions for Proposition 2 in Bourguignon and Chakravarty (2002). However, though these authors consider poverty indices based on the "union" approach of poverty identification, the set of indices they consider is characterized by a non-positive cross derivative $\pi^{(1,2)}(x_1, x_2)$—correlation increasing switches are then supposed not to raise the level of poverty—and complies with the strong focus and continuity axioms. Here, we impose correlation increasing switches not to decrease poverty and indices comply with the weak focus and restricted continuity axioms.

$$F_{\lambda,2}(x_2) = \begin{cases} F(x_2) & \text{if } x_2 \leq \min\{z_1, z_2\} \\ F(z_2) + P^{1,1}(z_1, x_2) - P^{1,1}(z_1, z_2) & \text{otherwise} \end{cases}. \tag{16}$$

In the more limiting case we impose $z_1 = z_2$, it can then easily be seen from the above equations that condition (14) boils down to traditional unidimensional first-order dominance conditions since we only needs to compare the cdf at each period up to the maximum value chosen deprivation line $z_t^+ \ \forall t \in \{1, 2\}$.

The opposite definition of the poverty domain, namely the "intersection" definition, supposes that individual are poor whenever they endure deprivations at every period, that is $\Gamma(\lambda) = \left\{(x_1, x_2) \in \Re_+^2 | \max\left\{\frac{x_1}{z_1}, \frac{x_2}{z_2}\right\} \leq 1\right\}$. This approach is for instance used by Tsui (2002). If we consider only those members of $\Pi(\lambda^+)$ so that the poverty domain fit the "intersection" definition, it can easily be seen that conditions in Theorem 1 and 2 come down to condition (12), *i.e.* comparisons of "intersection" headcount indices over the maximum "intersection" poverty domain. Tractability is thus garanteed. However, it can be noticed that in this case, there is no gain any more in imposing a lower bound for the poverty frontier since the dominance conditions are exactly the same.

Instead of considering a unique type of poverty frontier, it is possible to test robustness using a whole familly of poverty frontiers so that conditions (13) and (14) could be easily implemented. For instance, can assume that the poverty frontier can be defined using the arithmetic mean or some generalized mean as in Foster and Santos (2013). In that case, the function λ is a monotone transform of functions $\lambda_{\beta,z}$ defined as:

$$\lambda_{\beta,z}(x_1, x_2) := \begin{cases} \left(\frac{1}{2}\sum_{t=1}^{2} x_t^{\beta}\right)^{\frac{1}{\beta}} - z & \text{if } \beta \neq 0 \\ \left(\prod_{t=1}^{2} x_t^{\frac{1}{2}}\right) - z & \text{if } \beta = 0 \end{cases}, \tag{17}$$

It then becomes possible to implement conditions (13) and (14) for different values of β and z within appropriate ranges. The cases $\beta \to -\infty$ and $\beta \to +\infty$ respectively corresponds to the traditional "union" and "intersection" views of poverty identification, with $z_1 = z_2$, investigated in the previous paragraphs.

3.3 Symmetry and asymmetry

Symmetry and asymmetry are introduced Bresson and Duclos (2012) to deal with situations where an ordering of poor income profiles can be made when compared with their symmetric image with respect to the line of perfect equality. In an intertemporal setting, symmetry is a strong assumption as it means that the social evaluator is indifferent to the period at which incomes are enjoyed (after possibly adjusting for price differences and discounting preferences). Asymmetry is a more general hypothesis that assumes for instance that income profiles from $\Gamma_1(\lambda)$ are systematically preferred than their symmetric image in $\Gamma_2(\lambda)$, or vice versa.

Let $\mathcal{L}_S$ be the subset of $\mathcal{L}$ whose members are symmetric and consider the class Π_S of bidimensional symmetric poverty measures defined as:

$$\Pi_S(\lambda_S^+) = \left\{ P(\lambda) \in \Pi(\lambda_S^+) \;\middle|\; \begin{array}{l} \lambda \in \mathcal{L}_S, \\ \pi(x_1, x_2; \lambda_S) = \pi(x_2, x_1; \lambda_S), \forall (x_1, x_2) \in \Gamma(\lambda_S) \end{array} \right\}. \tag{18}$$

where $\lambda_S \in \mathcal{L}_S$ denotes a symmetric well-being-function.

The additional restriction imposed on π to define Π_S entails that the marginal effect of an income increment at the first period equals the marginal effect of the same increment at the second period, for two symmetric income profiles ($\pi^{(1)}(x_1, x_2; \lambda_S) = \pi^{(2)}(x_2, x_1; \lambda_S)$, $\forall (x_1, x_2) \in \Gamma(\lambda_S)$). In a similar manner, the variation of the marginal contribution of an income increment is symmetric for symmetric income profiles ($\pi^{(1,2)}(x_1, x_2; \lambda_S) = \pi^{(1,2)}(x_2, x_1; \lambda_S)$, $\forall (x_1, x_2) \in \Gamma(\lambda_S)$). Bresson and Duclos (2012) showed that adding symmetry to the basic set of properties considered by Duclos et al. (2006) dramatically improves the ordering power of the first-order dominance test since it makes it possible for relatively higher values of the cdf to be compensated by relatively lower values at the corresponding symmetric income profiles when comparing two distributions.

Theorem 3.

$$P_A(\lambda) \geq P_B(\lambda), \; \forall P(\lambda) \in \Pi_S(\lambda_S^+), \tag{19}$$

$$\textit{iff} \quad P_A^{1,1}(x_1, x_2) + P_A^{1,1}(x_2, x_1) \geq P_B^{1,1}(x_1, x_2) + P_B^{1,1}(x_2, x_1), \; \forall (x_1, x_2) \in \Gamma_1(\lambda_S^+) \tag{20}$$

$$\textit{and} \quad H_A(\lambda) \geq H_B(\lambda) \; \forall \lambda \in \mathcal{L}_S \; s.\; t.\; \Gamma(\lambda) \subseteq \Gamma(\lambda_S^+) \tag{21}$$

$$\textit{and} \quad \sum_{t=1}^{2} F_{\lambda,t}^A(x) \geq \sum_{t=1}^{2} F_{\lambda,t}^B(x) \; \forall x \in [0, z^*], \; \lambda \in \mathcal{L}_S \; s.\; t.\; \Gamma(\lambda) \subseteq \Gamma(\lambda_S^+). \tag{22}$$

Theorem 4.

$$P_A(\lambda) \geq P_B(\lambda), \; \forall P(\lambda) \in \Pi_S(\lambda_S^+), \; \lambda \in \mathcal{L}_S \; s.\; t.\; \Lambda(\lambda) \subseteq \bar{\Gamma}(\lambda_S^+, \lambda_S^-), \tag{23}$$

$$\textit{iff} \quad P_A^{1,1}(x_1, x_2) + P_A^{1,1}(x_2, x_1) \geq P_B^{1,1}(x_1, x_2) + P_B^{1,1}(x_2, x_1), \; \forall (x_1, x_2) \in \Gamma_1(\lambda_S^+) \tag{24}$$

$$\textit{and} \quad H_A(\lambda) \geq H_B(\lambda) \; \forall \lambda \in \mathcal{L}_S \; s.\; t.\; \Lambda(\lambda) \subseteq \bar{\Gamma}(\lambda_S^+, \lambda_S^-) \tag{25}$$

$$\textit{and} \quad \sum_{t=1}^{2} F_{\lambda,t}^A(x) \geq \sum_{t=1}^{2} F_{\lambda,t}^B(x) \; \forall x \in [0, z^*], \; \lambda \in \mathcal{L}_S \; s.\; t.\; \Lambda(\lambda) \subseteq \bar{\Gamma}(\lambda_S^+, \lambda_S^-). \tag{26}$$

Proof. See Appendix A. □

Theorem 3 (4) is the counterpart of Theorem 1 (2) for symmetric poverty indices. As in Bresson and Duclos (2012), we can observe that the ordering power should raise has we may observe compensations effects on both sides of the diagonal of perfect equality. The increase is also due to the use of symmetric poverty frontiers that make conditions 21 (25) and 22 (26) less demanding than with the general framework.

Symmetry may not be appropriate, however, in those cases in which we may not be (individually or socially) indifferent to a permutation of periodic incomes. We may yet feel that poverty is higher with income profile (x_1, x_2) than with (x_2, x_1) whenever $x_1 < x_2$. For instance, it may be deemed that low income may be more detrimental to well-being during childhood than during adulthood, since low income as a child can lead to poorer health and lower educational outcomes over the entire lifetime. Asymmetry is the general case in the class of intertemporal poverty indices proposed by Calvo and Dercon (2009) and Hoy and Zheng (2011), where weights at each period decrease as the final period is approached. A converse approach may also be suggested on the basis of the loss aversion argument that is central in Kahneman and Tversky's prospect theory (Günther and Maier, 2014, Jäntti, Kanbur, Nyyssölä and Pirttilä, 2014).

Without loss of generality, we assume that income profiles within $\Gamma_1(\lambda)$ never yield less poverty than their symmetric image in $\Gamma_2(\lambda)$, so that symmetry can be regarded as a limiting case of asymmetry. Contrary to Bresson and Duclos (2012), we suppose that the shape of the poverty frontier is symmetric with respect to the diagonal of perfect equality, though restricted continuity makes it possible for a given profile and its symmetric counterpart on the frontier not to show the same poverty level. While this restriction may be regarded as debatable, it is worth noting that most intertemporal indices are based on symmetric poverty frontiers.[7] We can then consider the following class of asymmetric monotone poverty measures:

$$\Pi_{AS}(\lambda_S^+) = \left\{ P(\lambda) \in \Pi(\lambda_S^+) \left| \begin{array}{l} \lambda \in \mathcal{L}_S, \\ \pi^{(x_1)}\big(x_1, z_2(x_1)\big) \leq \pi^{(x_1)}\big(z_1(x_1), x_1\big) \leq 0, \ \forall x_1 \in [0, z^*], \\ \pi^{(1)}(x_1, x_2; \lambda) \leq \pi^{(2)}(x_2, x_1; \lambda) \quad \text{if } x_1 \leq x_2, \\ \pi^{(1,2)}(x_1, x_2; \lambda) \geq \pi^{(1,2)}(x_2, x_1; \lambda) \quad \text{if } x_1 \leq x_2. \end{array} \right. \right\} \tag{27}$$

The first line to the right of (27) indicates the above mentioned restriction on the shape of the poverty frontier. The second means that moving towards the permanent income profile (z^*, z^*) along the poverty frontier results in . The third condition in the definition of $\Pi_{AS}(\lambda_S^+)$ implies that changes in the lowest income have a greater impact on poverty when the lowest income is the first period income. Consequently, for equal values of x_1 and x_2, changes in x_1 have a greater impact on welfare than changes in x_2. Finally, the fourth line states that, considering two symmetric income profiles, the marginal poverty benefit of an increase in either x_1 or x_2 decreases the most with the value of the other variable when the income profile is the one with the lowest first period income. It also says that a correlation decreasing switch decreases poverty more when x_1 is lower, for the same total income. Both lines emphasize the greater normative importance of those with lower first-period incomes.

Proposition 5.

$$P_A(\lambda) > P_B(\lambda), \ \forall P(\lambda) \in \Pi_{AS}(\lambda_S^+), \tag{28}$$

$$\textit{iff} \quad P_A^{1,1}(x_1, x_2) > P_B^{1,1}(x_1, x_2), \ \forall (x_1, x_2) \in \Gamma_1(\lambda_S^+) \tag{29}$$

[7] It deserves to be noticed that the remark holds for most multi-period indices that consists in the aggregation of censored relative poverty gaps $\min\left\{\frac{z_t - x_t}{z_t}, 1\right\}$. Though the deprivation lines z_t may differ from period to period, rescaling income series and deprivation makes it possible to obtain a symmetric poverty frontier without altering the estimated level of poverty. In this sense, we can argue that these measures are using symmetric poverty frontiers.

$$\text{and} \quad P_A^{1,1}(x_1,x_2) + P_A^{1,1}(x_2,x_1) > P_B^{1,1}(x_1,x_2) + P_B^{1,1}(x_2,x_1),\ \forall (x_1,x_2) \in \Gamma_1(\lambda_S^+) \tag{30}$$

$$\text{and} \quad H_A(\lambda) \geq H_B(\lambda)\ \forall \lambda \in \mathcal{L}_S\ s.\ t.\ \Gamma(\lambda) \subseteq \Gamma(\lambda_S^+) \tag{31}$$

$$\text{and} \quad \sum_{t=1}^{T} F_{\lambda,t}^{A}(x) \geq \sum_{t=1}^{T} F_{\lambda,t}^{B}(x)\ \forall x \in [0,z^*],\ T \in \{1,2\},\ \lambda \in \mathcal{L}_S\ s.\ t.\ \Gamma(\lambda) \subseteq \Gamma(\lambda_S^+). \tag{32}$$

Theorem 6.

$$P_A(\lambda) \geq P_B(\lambda),\ \forall P(\lambda) \in \Pi_{AS}(\lambda_S^+),\ \lambda \in \mathcal{L}_S\ s.\ t.\ \Lambda(\lambda) \subseteq \bar{\Gamma}(\lambda_S^+,\lambda_S^-), \tag{33}$$

$$\text{iff} \quad P_A^{1,1}(x_1,x_2) > P_B^{1,1}(x_1,x_2),\ \forall (x_1,x_2) \in \Gamma_1(\lambda_S^+) \tag{34}$$

$$\text{and} \quad P_A^{1,1}(x_1,x_2) + P_A^{1,1}(x_2,x_1) \geq P_B^{1,1}(x_1,x_2) + P_B^{1,1}(x_2,x_1),\ \forall (x_1,x_2) \in \Gamma_1(\lambda_S^+) \tag{35}$$

$$\text{and} \quad H_A(\lambda) \geq H_B(\lambda)\ \forall \lambda \in \mathcal{L}_S\ s.\ t.\ \Lambda(\lambda) \subseteq \bar{\Gamma}(\lambda_S^+,\lambda_S^-) \tag{36}$$

$$\text{and} \quad \sum_{t=1}^{T} F_{\lambda,t}^{A}(x) \geq \sum_{t=1}^{T} F_{\lambda,t}^{B}(x)\ \forall x \in [0,z^*],\ T \in \{1,2\},\ \lambda \in \mathcal{L}_S\ s.\ t.\ \Lambda(\lambda) \subseteq \bar{\Gamma}(\lambda_S^+,\lambda_S^-). \tag{37}$$

Proof. See Appendix A. ☐

As $\Pi_S(\lambda_S^+) \subset \Pi_{AS}(\lambda_S^+)$, the dominance conditions described in Theorem 5 (6) logically show a lower ordering power than the conditions suggested in Theorem 3 (4) for symmetric indices. The lower ordering power is due to conditions (29) and (32) (resp. (34) and (37)) that, in the comparison of populations A and B, higher values for population B of the corresponding headcount index defined with respect to income profiles in $\Gamma_1(\lambda_S^+)$ cannot be compensated by lower values of the same headcount index defined with respect to symmetric profiles from $\Gamma_2(\lambda_S^+)$ whereas the opposite is possible.

4 Chronic poverty orderings

As noted in the introduction, the results presented in the previous section are also relevant for the study of chronic poverty. However, chronic poverty is a rather complex notion and many authors have suggested different definitions of this concept. For instance, **?** proposed to define chronic poverty with respect to the duration of poverty spell while Rodgers and Rodgers (1993) identified the chronic poor as those individuals whose permanent income is below some given threshold.

Let suppose that intertemporal can be additively decomposed into chronic and transient components, that is $P(\lambda) = C\big(P(\lambda)\big) + T\big(P(\lambda)\big)$ where $C()$ and $T()$ respectively refers to the chronic and transient components. Additivity is of course an appealing property as it makes it easier to identify the main sources of poverty. It is worth emphasizing that this hypothesis has a very general character since the additive form is compatible with different approaches for the definition of chronic poverty. In the present section, we consider two of them, namely the "spell" and "components" approaches.

The "spell" approach for the chronic/transient poverty distinction basically supposes that individuals are either chronic poor or transient poor depending on their income profile. More

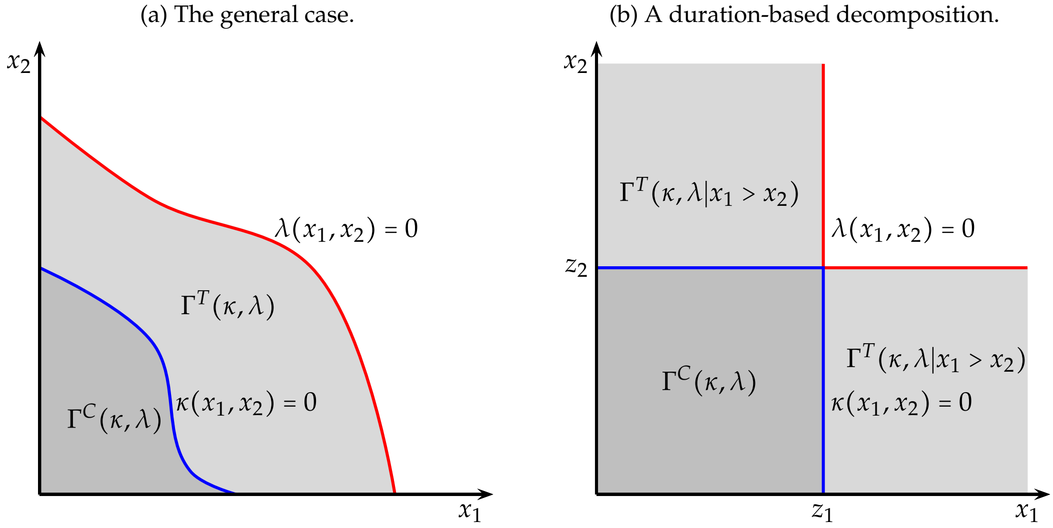

Figure 3: The chronic poverty domain as part of the intertemporal poverty domain with the "spell" approach.

specifically, the poverty domain $\Gamma(\lambda)$ can be split into two subsets that refer to each type of poor. Let Γ^C be the subset corresponding to the chronic poor and Γ^T denote the subset for the transient poor, with $\Gamma = \Gamma^C \cup \Gamma^T$ and $\Gamma^C \cap \Gamma^T = \emptyset$. Separating the chronic poor from the transient poor consequently entails the use of frontiers within the poverty domain. As for the poverty frontier, these frontiers are supposed to be implicitly defined by a locus of the form $\kappa(x_1, x_2) = 0$ with $\kappa : \Re^2_+ \to \Re$ being some non-decreasing continuous function. Chronic poverty is generally assumed to be associated with the less desirable situations in terms of welfare. As a consequence, we can also reasonably put forward that the worst profile $(0, 0) \in \Gamma^C$ so that, associated with the non-decreasingness of κ, Γ^C can be defined as the set of income profiles below the chronic poverty frontier, that is:

$$\Gamma^C(\kappa, \lambda) := \{(x_1, x_2) \in \Gamma(\lambda) | \kappa(x_1, x_2) \leq 0\}, \quad (38)$$

$$\Gamma^T(\kappa, \lambda) := \{(x_1, x_2) \in \Gamma(\lambda) | \kappa(x_1, x_2) \geq 0 \wedge \lambda(x_1, x_2) \leq 0\} \quad (39)$$

For instance, the chronic poverty frontier can be defined with respect to mean income as in Jalan and Ravallion (2000) or some generalized mean as in Foster and Santos (2013). In that case, we have $\kappa(x_1, x_2) = \lambda_{\beta,z}(x_1, x_2)$. Chronic poverty is also sometime defined with respect to the time spent with insufficient resources, that is if a person's income is below some deprivation line in s out of T time periods as in **?** or more recently Foster (2009). In that situation, the chronic poverty domain is defined with respect to monotone transform of functions of the type:

$$\kappa_s(x_1, x_2) = s - \sum_{t=1}^{2} \mathbb{I}(x_t < z_t), \quad (40)$$

where $\mathbb{I}(a) = 1$ if a in parenthesis is true, otherwise $\mathbb{I}(a) = 0$, and $z_t \in \Re_{++}$ is a time-specific deprivation line.

Figure 3 illustrates this partition of the poverty domain in the general case (left panel) as well as in the case of duration-based definitions of chronic poverty as in Foster (2009) (right panel).

A very different view of the chronic/transient poverty distinction is made with the "component" approach. This view states that poor individuals are likely to experience both chronic and transient forms of poverty during the whole period. Here, the chronic component of poverty is associated with the value of some intertemporal representive income while transient poverty is due to income variability during the period of interest. Consequently, the subsets of the poverty domain where each type of poverty is likely to be observed may intersect. For instance, if we assume that intertemporal poverty is the sum of a usual snapshot poverty index where current income is replaced by the average intertemporal income (*i.e.* the chronic component of poverty) and an additional element based on income variability (the transient component), then $\Gamma^T = \{(x_1, x_2) \in \Gamma(\lambda) | x_1 \neq x_2\}$ while the chronic poverty domain is defined as $\Gamma^C(\lambda) = \{(x_1, x_2) \in \Gamma(\lambda) | \frac{x_1+x_2}{2} \leq z^*\}$. In this situation, the two domains overlap, but we can observe that the chronic poverty domain shares many points with the type of chronic poverty domains stemming from the "spell" approach.

Indeed, with both approaches, we can observe that chronic poverty is measured on the bottom part of the poverty domain. More formally, we assume that Γ^C is defined so that $(0,0) \in \Gamma^C$ and $\Re^2_+ \not\pitchfork \Gamma^C$ is connected (*i.e.* there is no "hole" in Γ^C). The assumption that intertemporal poverty can be additively decomposed into chronic and transient parts makes it possible to express the chronic component of poverty as:

$$C\big(P(\lambda); \kappa\big) := \iint_{\Gamma^C(\kappa,\lambda)} \pi^c(x_1, x_2; \lambda)\, dF(x_1, x_2), \tag{41}$$

where π^c is a chronic poverty index that fulfils the same restrictions as π so that $C\big(P(\lambda); \kappa\big)$ becomes a member of $\Pi(\lambda^+)$. Considering the chronic component of intertemporal poverty as a multidimensional or intertemporal poverty index with a potential discontinuity at the poverty line then makes it possible to use Theorem 1 to 6 to test the robustness of chronic poverty orderings.

5 Conclusion

This paper suggests new procedures to test the ethical robustness of multidimensional or intertemporal poverty orderings as well as of chronic components of poverty rankings. Unlike previous contributions, notably Bourguignon and Chakravarty (2003), Duclos et al. (2006) and Bresson and Duclos (2012), this paper does not consider uniquely continuous poverty indices, but the wider set of indices that comply with restricted continuity, hence allowing for a discontinuity at the frontier of the poverty domain. This should not be regarded as a trifling difference because many poverty indices exhibit such discontinuities so that we are now able to provide dominance conditions for many multidimensional/intertemporal/chronic poverty indices proposed and applied in the related literatures. Dominance conditions are notably valid for Alkire and Foster's (2011) famous family of multidimensional poverty indices that generalizes Foster, Greer and Thorbecke (1984) family of undimensional poverty indices, though the conditions do not apply to the adjusted headcount index that serves as a basis for UNDP's MPI (Multidimensional Poverty Index).

This last remark highlights one of the crucial issues that still have to be addressed regarding multidimensional/intertemporal poverty orderings. Such indices like the adjusted headcount index or the intertemporal poverty indices where the persistence in poverty, measured by the contiguity of deprivations, matters, show discontinuities within the poverty domain. Further research should then identify the corresponding dominance conditions and try to check whether relaxing the restricted continuity assumption still makes it possible to test the robustness of poverty rankings with respect to both the location of the poverty frontier and the functional form of the poverty index.

A Proofs

The proofs rely extensively on Duclos et al. (2006) and Bresson and Duclos (2012).

We can define a two-period poverty index as a sum of low x_2 (with respect to x_1) and of low x_1 (with respect to x_2) poverty:

$$P(\lambda) = \int_0^{z^*} \int_{x_2}^{z_1(x_2)} \pi(x_1, x_2, \lambda) f(x_1, x_2) dx_1 dx_2 + \int_0^{z^*} \int_{x_1}^{z_2(x_1)} \pi(x_1, x_2; \lambda) f(x_1, x_2) dx_2 dx_1. \tag{42}$$

We first proceed with the first part of the right-hand term of (42). Integrating that expression by parts with respect to x_1, we find:

$$\begin{aligned}
\int_0^{z^*} \int_{x_2}^{z_1(x_2)} \pi(x_1, x_2, \lambda) f(x_1, x_2)\, dx_1 dx_2 &= \int_0^{z^*} \left[\pi(x_1, x_2) F(x_1|x_2)\right]_{x_1=x_2}^{x_1=z_1(x_2)} f(x_2)\, dx_2 \\
&\quad - \int_0^{z^*} \int_{x_2}^{z_1(x_2)} \pi^{(1)}(x_1, x_2) F(x_1|x_2) f(x_2)\, dx_1 dx_2 && (43) \\
&= \int_0^{z^*} \pi\big(z_1(x_2), x_2\big) F\big(z_1(x_2)|x_2\big) f(x_2)\, dx_2 \\
&\quad - \int_0^{z^*} \pi(x_2, x_2) F(x_1 = x_2|x_2) f(x_2)\, dx_2 \\
&\quad - \int_0^{z^*} \int_{x_2}^{z_1(x_2)} \pi^{(1)}(x_1, x_2) F(x_1|x_2) f(x_2)\, dx_1 dx_2. && (44)
\end{aligned}$$

Regarding the first element of the right-hand term of (44), integration by parts yields:

$$\begin{aligned}
\int_0^{z^*} \pi\big(z_1(x_2), x_2\big) F\big(z_1(x_2)|x_2\big) f(x_2)\, dx_2 &= \left[\pi\big(z_1(x_2), x_2\big) F_{\lambda,2}(x_2)\right]_0^{z^*} \\
&\quad - \int_0^{z^*} \pi^{(x_2)}\big(z_1(x_2), x_2\big) F_{\lambda,2}(x_2)\, dx_2 && (45) \\
&= \pi(z^*, z^*) F_{\lambda,2}(z^*) \\
&\quad - \int_0^{z^*} \pi^{(x_2)}\big(z_1(x_2), x_2\big) F_{\lambda,2}(x_2)\, dx_2, && (46)
\end{aligned}$$

since $F_{\lambda,2}(0) = 0$.

To integrate the third element of the right-hand term of (44) by parts with respect to x_2, let $K(x_2) = \int_{x_2}^{z_1(x_2)} \pi^{(1)}(x_1, x_2)\, F(x_1, x_2)\, dx_1$. We then get:

$$\begin{aligned}
\frac{\partial K(x_2)}{\partial x_2} &= z_1'(x_2) \pi^{(1)}\big(z_1(x_2), x_2\big) F\big(z_1(x_2), x_2\big) - \pi^{(1)}(x_2, x_2) F(x_2, x_2) \\
&\quad + \int_{x_2}^{z_1(x_2)} \pi^{(1,2)}(x_1, x_2) F(x_1, x_2)\, dx_1 + \int_{x_2}^{z_1(x_2)} \pi^{(1)}(x_1, x_2) F(x_1|x_2) f(x_2)\, dx_1. && (47)
\end{aligned}$$

Integrating that expression along x_2 and over $[0, z^*]$ and rearranging, we have:

$$\begin{aligned}
&\int_0^{z^*} \int_{x_2}^{z_1(x_2)} \pi^{(1)}(x_1, x_2) F(x_1|x_2) f(x_2)\, dx_1 dx_2 && (48) \\
&= \left[K(x_2)\right]_0^{z^*} - \int_0^{z^*} z_1'(x_2) \pi^{(1)}\big(z_1(x_2), x_2\big) F\big(z_1(x_2), x_2\big)\, dx_2 \\
&+ \int_0^{z^*} \pi^{(1)}(x_2, x_2) F(x_2, x_2)\, dx_2 - \int_0^{z^*} \int_{x_2}^{z_1(x_2)} \pi^{(1,2)}(x_1, x_2) F(x_1, x_2)\, dx_1 dx_2 && (49) \\
&= -\int_0^{z^*} z_1'(x_2) \pi^{(1)}\big(z_1(x_2), x_2\big) F\big(z_1(x_2), x_2\big)\, dx_2 + \int_0^{z^*} \pi^{(1)}(x_2, x_2) F(x_2, x_2)\, dx_2
\end{aligned}$$

$$- \int_0^{z^*} \int_{x_2}^{z_1(x_2)} \pi^{(1,2)}(x_1, x_2) F(x_1, x_2)\, dx_1 dx_2, \tag{50}$$

since $z_1(z^*) = z^*$ (hence $K(z^*) = 0$) and $F(x_1, 0) = 0\ \forall x_1$ (hence $K(0) = 0$). Using (44), (46) and (50), we obtain:

$$\begin{aligned}
&\int_0^{z^*} \int_{x_2}^{z_1(x_2)} \pi(x_1, x_2, \lambda) f(x_1, x_2) dx_1 dx_2 \\
&= \pi(z^*, z^*) F_{\lambda,2}(z^*) - \int_0^{z^*} \pi^{(x_2)}\big(z_1(x_2), x_2\big) F_{\lambda,2}(x_2)\, dx_2 \\
&- \int_0^{z^*} \pi(x_2, x_2) F(x_1 = x_2 | x_2) f(x_2)\, dx_2 + \int_0^{z^*} z_1'(x_2) \pi^{(1)}\big(z_1(x_2), x_2\big) F\big(z_1(x_2), x_2\big)\, dx_2 \\
&- \int_0^{z^*} \pi^{(1)}(x_2, x_2) F(x_2, x_2)\, dx_2 + \int_0^{z^*} \int_{x_2}^{z_1(x_2)} \pi^{(1,2)}(x_1, x_2) F(x_1, x_2)\, dx_1 dx_2.
\end{aligned} \tag{51}$$

Proceeding similarly with the second part of the right-hand term of (42) and adding the above, we obtain:

$$\begin{aligned}
P(\lambda) &= \pi(z^*, z^*) F_{\lambda,2}(z^*) - \int_0^{z^*} \pi^{(x_2)}\big(z_1(x_2), x_2\big) F_{\lambda,2}(x_2)\, dx_2 \\
&- \int_0^{z^*} \pi(x_2, x_2) F(x_1 = x_2 | x_2) f(x_2)\, dx_2 + \int_0^{z^*} z_1'(x_2) \pi^{(1)}\big(z_1(x_2), x_2\big) F\big(z_1(x_2), x_2\big)\, dx_2 \\
&- \int_0^{z^*} \pi^{(1)}(x_2, x_2) F(x_2, x_2)\, dx_2 + \int_0^{z^*} \int_{x_2}^{z_1(x_2)} \pi^{(1,2)}(x_1, x_2) F(x_1, x_2)\, dx_1 dx_2 \\
&+ \pi(z^*, z^*) F_{\lambda,1}(z^*) - \int_0^{z^*} \pi^{(x_1)}\big(x_1, z_2(x_1)\big) F_{\lambda,1}(x_1)\, dx_1 \\
&- \int_0^{z^*} \pi(x_1, x_1) F(x_2 = x_1 | x_1) f(x_1)\, dx_1 + \int_0^{z^*} z_2'(x_1) \pi^{(2)}\big(x_1, z_2(x_1)\big) F\big(x_1, z_2(x_1)\big)\, dx_1 \\
&- \int_0^{z^*} \pi^{(2)}(x_1, x_1) F(x_1, x_1)\, dx_1 + \int_0^{z^*} \int_{x_1}^{z_2(x_1)} \pi^{(1,2)}(x_1, x_2) F(x_1, x_2)\, dx_2 dx_1
\end{aligned} \tag{52}$$

It can be observed that $F(x_2 = x_1 | x_1) f(x_1) = \frac{\partial F(x_1, x_1)}{\partial x_1} - F(x_1 | x_2 = x_1) f(x_2 = x_1)$, so that:

$$\begin{aligned}
&\int_0^{z^*} \pi(x_1, x_1) F(x_2 = x_1 | x_1) f(x_1)\, dx_1 \\
&= \int_0^{z^*} \pi(x_1, x_1) \frac{\partial F(x_1, x_1)}{\partial x_1}\, dx_1 - \int_0^{z^*} \pi(x_1, x_1) F(x_1 | x_2 = x_1) f(x_2 = x_1)\, dx_1 &(53)\\
&= [\pi(x_1, x_1) F(x_1, x_1)]_0^{z^*} - \int_0^{z^*} \Big(\pi^{(1)}(x_1, x_1) + \pi^{(2)}(x_1, x_1)\Big) F(x_1, x_1)\, dx_1 \\
&- \int_0^{z^*} \pi(x_1, x_1) F(x_1 | x_2 = x_1) f(x_2 = x_1)\, dx_1 &(54)\\
&= \pi(z^*, z^*) F(z^*, z^*) - \int_0^{z^*} \Big(\pi^{(1)}(x_1, x_1) + \pi^{(2)}(x_1, x_1)\Big) F(x_1, x_1)\, dx_1 \\
&- \int_0^{z^*} \pi(x_2, x_2) F(x_1 = x_2 | x_2) f(x_2)\, dx_2. &(55)
\end{aligned}$$

Using this two last result, changing the integration variable in $\int_0^{z^*} \pi^{(2)}(x_2, x_2) F(x_2, x_2)\ dx_2$ and rearranging, we then have:

$$\begin{aligned}
P(\lambda) &= \pi(z^*, z^*)\big(F_{\lambda,1}(z^*) + F_{\lambda,2}(z^*) - F(z^*, z^*)\big) \\
&- \int_0^{z^*} \pi^{(x_2)}\big(z_1(x_2), x_2\big) F_{\lambda,2}(x_2)\, dx_2
\end{aligned}$$

$$
\begin{aligned}
&- \int_0^{z^*} \pi^{(x_1)}\big(x_1, z_2(x_1)\big) F_{\lambda,1}(x_1)\, dx_1 \\
&+ \int_0^{z^*} z_1'(x_2)\pi^{(1)}\big(z_1(x_2), x_2\big) F\big(z_1(x_2), x_2\big)\, dx_2 \\
&+ \int_0^{z^*} z_2'(x_1)\pi^{(2)}\big(x_1, z_2(x_1)\big) F\big(x_1, z_2(x_1)\big)\, dx_1 \\
&+ \int_0^{z^*} \int_{x_2}^{z_1(x_2)} \pi^{(1,2)}(x_1, x_2) F(x_1, x_2)\, dx_1 dx_2 \\
&+ \int_0^{z^*} \int_{x_1}^{z_2(x_1)} \pi^{(1,2)}(x_1, x_2) F(x_1, x_2)\, dx_2 dx_1.
\end{aligned} \tag{56}
$$

The four last terms of the right-hand term of (56) corresponds to the case studied in Bresson and Duclos (2012) where π is zero at the poverty frontier. It can easily be seen that the quantity $\big(F_{\lambda,1}(z^*) + F_{\lambda,2}(z^*) - F(z^*, z^*)\big)$ is simply the value of the headcount index $H(\lambda)$ associated with the poverty domain $\Gamma(\lambda)$, and so is non-negative. The rest of the proof follows by inspection.

Symmetry implies the following properties:

$$z_1(x_2) = z_2(x_2) \quad \forall x_2, \tag{57}$$

$$\pi^{(1)}(x_1, x_2) = \pi^{(2)}(x_2, x_1) \quad \forall x_1, x_2, \tag{58}$$

$$\pi^{(1,2)}(x_1, x_2) = \pi^{(1,2)}(x_2, x_1) \quad \forall x_1, x_2, \tag{59}$$

$$\pi^{(x_2)}\big(z_1(x_2), x_2\big) = \pi^{(x_1)}\big(x_1, z_2(x_1)\big) \quad \forall x_1 = x_2. \tag{60}$$

Consequently, equation (56) can be rewritten as:

$$
\begin{aligned}
P(\lambda) = {} & \pi(z^*, z^*)\big(F_{\lambda,1}(z^*) + F_{\lambda,2}(z^*) - F(z^*, z^*)\big) \\
& - \int_0^{z^*} \pi^{(x_2)}\big(z_1(x_2), x_2\big) \left(F_{\lambda,1}(x_2) + F_{\lambda,2}(x_2)\right)\, dx_2 \\
& + \int_0^{z^*} z_1^{(1)}(x_2)\pi^{(1)}\big(z_1(x_2), x_2\big)\Big(F\big(z_1(x_2), x_2\big) + F\big(x_2, z_1(x_2)\big)\Big)\, dx_2 \qquad (61) \\
& + \int_0^{z^*} \int_{x_2}^{z_1(x_2)} \pi^{x_1,x_2}(x_1, x_2)\big(F(x_1, x_2) + F(x_2, x_1)\big)\, dx_1 dx_2 \qquad (62)
\end{aligned}
$$

In the present paper asymmetry implies the following properties:

$$z_1(x_2) = z_2(x_2) \quad \forall x_2, \tag{63}$$

$$\pi^{(1)}(x_1, x_2) \le \pi^{(2)}(x_2, x_1) \quad \forall (x_1, x_2) \in \Gamma_1(\lambda), \tag{64}$$

$$\pi^{(1,2)}(x_1, x_2) \ge \pi^{(1,2)}(x_2, x_1) \quad \forall (x_1, x_2) \in \Gamma_1(\lambda), \tag{65}$$

$$\pi^{(x_2)}\big(z_1(x_2), x_2\big) \le \pi^{(x_1)}\big(x_1, z_2(x_1)\big) \quad \forall x_1 = x_2. \tag{66}$$

Equation (56) then can be expressed:

$$
\begin{aligned}
P(\lambda) = {} & \pi(z^*, z^*)\big(F_{\lambda,1}(z^*) + F_{\lambda,2}(z^*) - F(z^*, z^*)\big) \\
& - \int_0^{z^*} \pi^{(x_1)}\big(z_2(x_1), x_1\big) \left(F_{\lambda,1}(x_1) + F_{\lambda,2}(x_1)\right)\, dx_1 \\
& - \int_0^{z^*} \left(\pi^{(x_1)}\big(x_1, z_2(x_1)\big) - \pi^{(x_1)}\big(z_2(x_1), x_1\big)\right) F_{\lambda,1}(x_1)\, dx_1 \\
& + \int_0^{z^*} z_2'(x_1)\pi^{(2)}\big(z_2(x_1), x_1\big)\left[F\big(z_2(x_1), x_1\big) + F\big(x_1, z_2(x_1)\big)\right]\, dx_1 \\
& + \int_0^{z^*} z_2'(x_1)\left[\pi^{(1)}\big(x_1, z_2(x_1)\big) - \pi^{(2)}\big(z_2(x_1), x_1\big)\right] F\big(x_1, z_2(x_1)\big)\, dx_1
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^{z^*}\int_{x_2}^{z_2(x_2)} \pi^{(1,2)}(x_1,x_2)\,[F(x_1,x_2)+F(x_2,x_1)]\,dx_1dx_2 \\
&+\int_0^{z^*}\int_{x_1}^{z_2(x_1)} \left[\pi^{(1,2)}(x_1,x_2)-\pi^{(1,2)}(x_2,x_1)\right]F(x_1,x_2)dx_2dx_1,. \qquad (67)
\end{aligned}
$$

The rest of the proof follows by inspection.

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